Symmetric Hochschild extension algebras and normalized 2-cocycles

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- K: an algebraically closed field
- Algebras mean bound quiver algebras over K
- $D = \mathrm{Hom}_K(-, K)$: standard duality

Definition

An extension of K-algebra A is a K-algebra epimorphism $\rho : T \to A$. An extension $\rho : T \to A$ of A is called *Hochschild extension* by a duality bimodule DA if $\operatorname{Ker}\rho \cong DA$ as T-bimodule. Then, T is called a *Hochschild extension algebra* of A by DA.

$T \cong A \oplus DA$ as K-vector space
Suppose that for $(a, f), (b, g) \in T \cong A \oplus DA$

$$(a, f)(b, g) = (ab, ag + fb + \alpha(a, b))$$

where $\alpha : A \times A \to DA$. Then α is a 2-cocycle.

Definition

A map $\alpha : A \times A \to DA$ is called a 2-*cocycle* if α is a bilinear map and α satisfies the following:

$$a\alpha(b, c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)c = 0$$

for any $a, b, c \in A$.

We denote the Hochschild extension algebra T of A by DA for 2-cocycle α by

$$T_\alpha(A, DA)$$

- $T_0(A, DA)$ is the trivial extension algebra of A by DA
- $\{(e, \sum_{e' \in Q_0} -\alpha(e, e')e') \mid e \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $T_\alpha(A, DA)$
- $\mathrm{H}^2(A, DA) \xleftrightarrow{1:1} \{$Hochschild extensions of A by $DA\}/\sim$
For any K-linear map $f : A \to DA$, $[\alpha - \delta f] = [\alpha]$, where δf is a 2-cocycle given by

$$(\delta f)(a, b) = af(b) - f(ab) + f(a)b$$

- $T_\alpha(A, DA)$ is self-injective.

Example 1. The trivial extension algebra $T_0(A, DA) = A \ltimes DA$ is symmetric. In fact, $T_0(A, DA)$ has a regular symmetric map $\mu_0 : T_0(A, DA) \to K$ given by

$$\mu_0(a, f) = f(1_A)$$

Example 2. Let $A = K[x, y]/(x^2, y^2)$, $B = \{1, x, y, xy\}$ a basis of A, $B^* = \{1^*, x^*, y^*, (xy)^*\}$ the dual basis of B and $\alpha : A \times A \to DA$ a 2-cocycle given by

$$\alpha(x, y) = 1^*$$
$$\alpha(a, b) = 0$$

for $(a, b) \in (B \times B) \setminus \{(x, y)\}$. Then $T_\alpha(A, DA)$ is not symmetric.

Question

When Hochschild extension algebras are symmetric?

Theorem [Ohnuki-Takeda-Yamagata (1999)]

Let $A = KQ/I$ and $\alpha : A \times A \to DA$ a 2-cocycle. If α satisfies

$$\alpha(p, q)(t(q)) = \alpha(q, p)(t(p))$$

for any two paths p, q of length 1 or more which pq is a cycle ($s(p) = t(q)$ and $t(p) = s(q)$), then $T_\alpha(A, DA)$ is symmetric.

Example 3. Let $A = K[x,y,z]/(x,y,z)^2$ and $B = \{1,x,y,z\}$ a basis of A, $B^* = \{1^*,x^*,y^*,z^*\}$ the dual basis of B and $\alpha : A \times A \to DA$ a 2-cocycle given by

$$\alpha(x,y) = 1^* - z^*$$
$$\alpha(y,z) = 1^* - x^*$$
$$\alpha(z,x) = 1^* - y^*$$
$$\alpha(a,b) = 0$$

for $(a,b) \in (B \times B)\backslash\{(x,y),(y,z),(z,x)\}$.
Then, $\alpha(x,y)(1) \neq \alpha(y,x)(1)$, however $T_\alpha(A,DA)$ is symmetric. In fact, $T_\alpha(A,DA)$ has a symmetric regular K-linear map $\lambda : T_\alpha(A,DA) \to K$ given by

$$\lambda(a,f) = f(1+x+y+z)$$

for $(a,f) \in T_\alpha(A,DA)$.

Theorem [I]

Let $A = KQ/I$ and B a basis of A which for each $b \in B$ b is represented by a path, $\alpha : A \times A \to DA$ a 2-cocycle,
$\eta_\alpha : A \times A \to DA$ a bilinear map given by $\eta_\alpha(a, b) = \alpha(a, b) - \alpha(b, a)$, and $V_\alpha = \{a \in Z(A) \mid f(a) = 0$ for any $f \in \eta_\alpha(A \times A)\}$.
If there exists $x \in V_\alpha$ such that $e^*(x) \neq 0$ for any vertices $e \in B$, then $T_\alpha(A, DA)$ is symmetric.

Proof. It is sufficient to check that a K-linear map $\lambda : T_\alpha(A, DA) \to K$ given by

$$\lambda(a, f) = f(x)$$

is regular and symmetric.

Example 4.

- In Example 1, $1 \in V_0$
- In Example 3, $V_\alpha = \langle 1 + x + y + z \rangle$.

Definition

Let A be a K-algebra, E a complete set of primitive orthogonal idempotents of A and $\alpha : A \times A \to DA$ a 2-cocycle. Then

- α is *normalized* if $\alpha(1, a) = \alpha(a, 1) = 0$ for any $a \in A$
- α is E*-normalized* if $\alpha(e, a) = \alpha(a, e) = 0$ for any $a \in A$ and $e \in E$

- $h_R(\alpha) : A \to DA$: a K-linear map given by

$$(h_R(\alpha))(a) = \sum_{e \in E} \alpha(a, e)e$$

- $h_L(\alpha) : A \to DA$: a K-linear map given by

$$(h_L(\alpha))(a) = \sum_{e \in E} e\alpha(e, a)$$

- $H_R(\alpha) = \alpha - \delta h_R(\alpha)$
- $H_L(\alpha) = \alpha - \delta h_L(\alpha)$

Then $H_L(H_R(\alpha)) = H_R(H_L(\alpha))$. We denote it by $\overline{\alpha}$.

Proposition

The following holds:

(1) $[\alpha] = [\overline{\alpha}]$ **(in particular,** $T_\alpha(A, DA) \cong T_{\overline{\alpha}}(A, DA)$**)**

(2) $\overline{\alpha}$ **is** E**-normalized**

(3) **Suppose that** $E = Q_0$**, then** $\overline{\alpha}$ **is given by**

$$\overline{\alpha}(p, q) = \begin{cases} s(p)\alpha(p, q)t(q) - p\alpha(t(p), s(q))q & \text{if } pq \neq 0 \text{ in } KQ, \\ 0 & \text{otherwise} \end{cases}$$

for paths p, q **in** A**.**

Remark. Suppose that α satisfies

$$\alpha(p,q)(t(q)) = \alpha(q,p)(t(p))$$

for any two paths p, q of length 1 or more which pq is a cycle ($s(p) = t(q)$ and $t(p) = s(q)$).

Let $\beta = \alpha - \delta h_\alpha$, where a K-linear map $h_\alpha : A \to DA$ is given by

$$h_\alpha(p) = \begin{cases} 0 & \text{if } p \text{ is a cycle in } Q \\ & \text{and the length of } p \text{ is 1 or more} \\ (h_R(\alpha))(p) & \text{otherwise} \end{cases}$$

for a path p in A.

Then, $1 \in V_{\overline{\beta}} = \{a \in Z(A) \mid f(a) = 0 \text{ for } f \in \eta_{\overline{\beta}}(A \times A)\}$.

Thank you for your attention!