Functor categories on derived categories of hereditary algebras

Yuta Kimura

Nagoya University

Notation

- k : alg. closed field
- algebra = finite dimensional k-algebra

A : algebra

- A-module = finite dimensional (left) A-module
  = finitely presented (left) A-module
- mod$A$: the category of A-modules

Motivation

Definition

A : algebra

- A : representation finite
  $:\Leftrightarrow \sharp\{$ iso. class of indec. A-module $\} < \infty$
  $\Leftrightarrow {}^{\exists}M \in \mathrm{mod}A$: basic s.t. $\mathrm{add}M = \mathrm{mod}A$
- A : rep. finite
  $\underline{\mathrm{Aus}}(A) := \mathrm{End}_A(M)/[\mathrm{proj}A]$
  : stable Auslander algebra of A

Motivation

A : hereditary algebra $:\Leftrightarrow \mathrm{gl.dim}(\mathsf{mod}\, A) \leq 1$

Theorem [Iyama-Oppermann]

If A is rep. finite hereditary algebra, then we have a triangle equivalence

$$\underline{\mathsf{mod}}\, \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A) \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \underline{\mathsf{Aus}}(A))$$

$\mathsf{mod}\, \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$

: the category of finitely presented $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$-modules (Frobenius category)

Today

Aim

Extend the equivalence

$$\underline{\mathrm{mod}}\, \mathrm{D^b}(\mathrm{mod} A) \simeq \mathrm{D^b}(\mathrm{mod}\, \underline{\mathrm{Aus}}(A))$$

for rep. infinite hereditary algebra A.

A : rep. infinite :⇔ not rep. finite

If A : rep. infinite, then $\underline{\mathrm{Aus}}(A)$ does not exist.

⇒ Take modules over categories.

Modules over categories

Example 1

For a ring R, define two categories $\mathcal{C}_R$ and $\operatorname{Mod}\mathcal{C}_R$:

- $\mathcal{C}_R$

  $\operatorname{Ob}\mathcal{C}_R := \{\bullet\}, \quad \mathcal{C}_R(\bullet, \bullet) := R^{\mathrm{op}}$

- $\operatorname{Mod}\mathcal{C}_R$

  Object $:=\{$ contra. additive functor $\mathcal{C}_R \to \mathcal{A}b$ $\}$

  Morphism $:=\{$ natural transformation $\}$

Then we have an equivalence:

$$\operatorname{Mod}\mathcal{C}_R \xrightarrow{\sim} \operatorname{Mod}R, \qquad M \mapsto M(\bullet)$$

Modules over categories

Example 2

For a finite acyclic quiver Q, define two categories:

- $\mathcal{C}_Q$

  $$\operatorname{Ob}\mathcal{C}_Q := Q_0$$

  $$\mathcal{C}_Q(i,j) := \mathsf{Vect}_k\{\text{ path } j \to i \text{ }\} \quad \text{for } i,j \in Q_0$$

- $\operatorname{Mod}\mathcal{C}_Q$

  $$\text{Object} := \{\text{ contra. additive functor } \mathcal{C}_Q \to \mathcal{A}b \text{ }\}$$

  $$\text{Morphism} := \{\text{ natural transformation }\}$$

Then we have an equivalence:

$$\operatorname{Mod}\mathcal{C}_Q \xrightarrow{\sim} \operatorname{Mod} kQ, \qquad M \mapsto \oplus_{i \in Q_0} M(i)$$

Modules over categories

Note that $\mathcal{C}_R, \mathcal{C}_Q$ are pre-additive categories.

Definition

$\mathcal{C}$: pre-additive cat.

- $\mathcal{C}$-module is a contra. additive functor $\mathcal{C} \to \mathcal{A}b$.
- $\operatorname{Mod}\mathcal{C}$
  - Object $:=\{$ $\mathcal{C}$-module $\}$
  - Morphism $:=\{$ natural transformation $\}$

- $\operatorname{Mod}\mathcal{C}$: abelian category
- $\mathcal{C}(-, X) \in \operatorname{Mod}\mathcal{C}$: projective for any $X \in \mathcal{C}$

Modules over categories

$\mathcal{C}$: additive category

Definition

$M \in \operatorname{Mod} \mathcal{C}$

- M: finitely presented if $\exists$ exact seq.

$$\mathcal{C}(-,X) \to \mathcal{C}(-,Y) \to M \to 0$$

  for some $X, Y \in \mathcal{C}$.

- $\operatorname{mod} \mathcal{C} := \{M \in \operatorname{Mod} \mathcal{C} \mid M \ : \text{finitely presented}\}$

- $\mathcal{T}$: triangulated $\Rightarrow$ $\operatorname{mod} \mathcal{T}$: Frobenius category
  $\Rightarrow$ $\underline{\operatorname{mod}}\, \mathcal{T}$: triangulated

Example. $\underline{\operatorname{mod}}\, \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$: triangulated

Modules over categories

Example

A : rep. finite algebra
$\underline{\mathsf{Aus}}(A)$: stable Auslander algebra
Then we have

$$\mathrm{mod}\,\underline{\mathsf{Aus}}(A) \simeq \mathrm{mod}(\underline{\mathrm{mod}}\,A)$$

If A is rep. finite hereditary algebra, then we have

$$\begin{aligned}\underline{\mathrm{mod}}\,\mathrm{D}^{\mathrm{b}}(\mathrm{mod}A) &\simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\underline{\mathsf{Aus}}(A)) \\ &\simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\underline{\mathrm{mod}}\,A))\end{aligned}$$

Modules over categories

Aim

For a rep. infinite hereditary algebra A, we show

$$\underline{\mathrm{mod}}\,\mathrm{D}^{\mathrm{b}}(\mathrm{mod}A) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\underline{\mathrm{mod}}\,A)).$$

We show two equivalences :

$$\underline{\mathrm{mod}}\,\mathrm{D}^{\mathrm{b}}(\mathrm{mod}A) \;\simeq\; \underline{\mathrm{mod}}(\widehat{\underline{\mathrm{mod}}\,A}) \;\simeq\; \mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\underline{\mathrm{mod}}\,A)).$$

$\widehat{\underline{\mathrm{mod}}\,A}$: repetitive category of $\underline{\mathrm{mod}}\,A$.

Repetitive categories

$\mathcal{C}$: k-linear additive category
$\mathrm{D} = \mathsf{Hom}_k(-, k)$

Definition

- $\widetilde{\mathcal{C}}$: k-linear pre-additive category defined by
$$\mathsf{Ob}\,\widetilde{\mathcal{C}} := \{(X, i) \mid X \in \mathcal{C}, i \in \mathbb{Z}\}$$
$$\widetilde{\mathcal{C}}((X,i),(Y,j)) := \begin{cases} \mathcal{C}(X,Y) & i = j, \\ \mathrm{D}\mathcal{C}(Y,X) & j = i+1, \\ 0 & \text{else.} \end{cases}$$
- Repetitive category $\widehat{\mathcal{C}}$ is a k-linear additive category generated by $\widetilde{\mathcal{C}}$.

First equivalence

A : hereditary algebra

$\mathbb{S} = \mathrm{D}A \otimes^{\mathbb{L}}_A - : \mathrm{D^b}(\mathrm{mod} A) \xrightarrow{\sim} \mathrm{D^b}(\mathrm{mod} A)$,

Theorem A [IO, K]

A : hereditary algebra

(a) We have an equivalence of additive categories

$$\widehat{\underline{\mathrm{mod}}\, A} \simeq \mathrm{D^b}(\mathrm{mod} A), \quad (X, i) \mapsto \mathbb{S}^i(X)$$

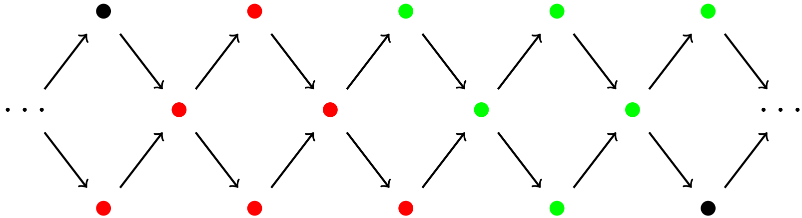
(b) We have a triangle equivalence

$$\underline{\mathrm{mod}}(\widehat{\underline{\mathrm{mod}}\, A}) \simeq \underline{\mathrm{mod}}\, \mathrm{D^b}(\mathrm{mod} A)$$

Image for (a) $\widehat{\underline{\mathrm{mod}}\, A} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod} A)$

$Q = [\bullet \to \bullet \to \bullet]$ and $A = kQ$: path algebra

AR-quiver of $\mathrm{D}^{\mathrm{b}}(\mathrm{mod} A)$

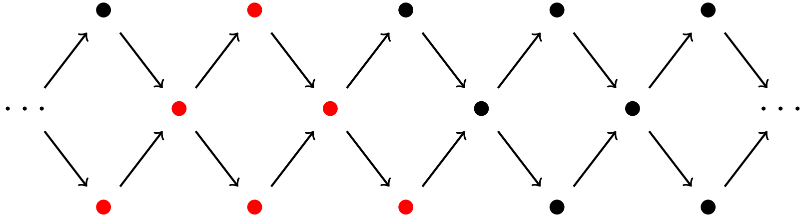


$\mathrm{mod}\, A$ $\quad$ $(\mathrm{mod}\, A)[1]$

Image for (a) $\widehat{\underline{\mathrm{mod}}\, A} \simeq \mathrm{D^b}(\mathrm{mod} A)$

$Q = [\bullet \to \bullet \to \bullet]$ and $A = kQ$: path algebra

AR-quiver of $\mathrm{D^b}(\mathrm{mod} A)$

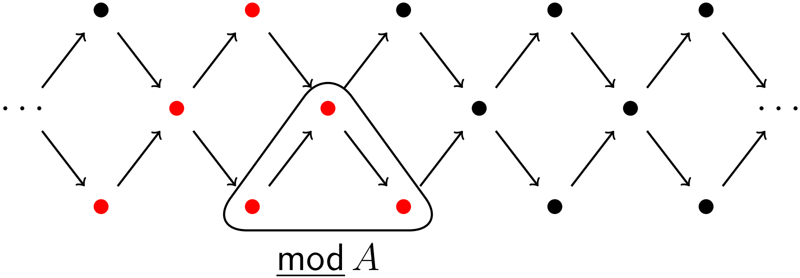


mod A

Image for (a) $\widehat{\underline{\mathrm{mod}}\, A} \simeq \mathrm{D^b}(\mathrm{mod}\, A)$

$Q = [\bullet \to \bullet \to \bullet]$ and $A = kQ$: path algebra

AR-quiver of $\mathrm{D^b}(\mathrm{mod}\, A)$

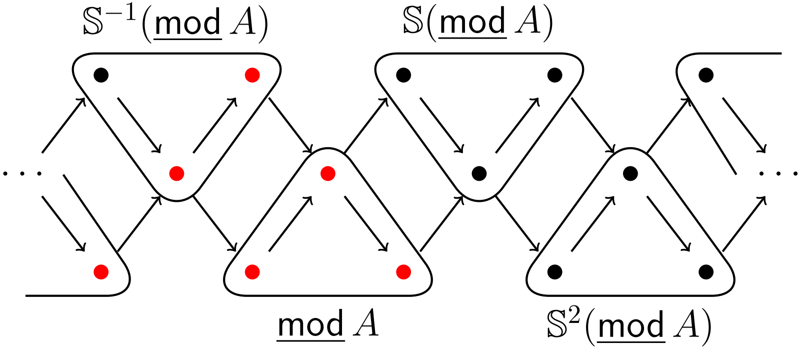


$\mathrm{mod}\, A$

Image for (a) $\widehat{\underline{\mathsf{mod}}\, A} \simeq \mathrm{D^b}(\mathrm{mod} A)$

$Q = [\bullet \to \bullet \to \bullet]$ and $A = kQ$: path algebra

AR-quiver of $\mathrm{D^b}(\mathrm{mod} A)$



mod A

Second equivalence

$$\underline{\mathrm{mod}}\, \mathrm{D}^{\mathrm{b}}(\mathrm{mod} A) \simeq \underline{\mathrm{mod}}(\widehat{\underline{\mathrm{mod}}\, A}) \qquad \text{done}$$

$$\simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}(\underline{\mathrm{mod}}\, A)) \qquad \text{next}$$

Strategy

- $\underline{\mathrm{mod}}\, A$: dualizing variety

In general, for a dulaizing variety $\mathcal{C}$ with some property, we have

$$\underline{\mathrm{mod}}\, \widehat{\mathcal{C}} \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, \mathcal{C}).$$

Dualizing variety

$\mathcal{C}$: k-linear additive category

We have a contra. functor

$$\mathrm{D} : \mathsf{Mod}\,\mathcal{C} \to \mathsf{Mod}\,\mathcal{C}^{\mathrm{op}}, \quad M \mapsto \mathrm{D}M$$

Definition [Auslander-Reiten]

$\mathcal{C}$: dualizing variety if

- $\mathcal{C}$ is idempotent complete and Hom-finite.
- $\mathrm{D} : \mathsf{Mod}\,\mathcal{C} \to \mathsf{Mod}\,\mathcal{C}^{\mathrm{op}}$ induces a duality $\mathsf{mod}\,\mathcal{C} \to \mathsf{mod}\,\mathcal{C}^{\mathrm{op}}$

Dualizing variety

- [Auslander-Reiten]

A : algebra $\Rightarrow$ proj$A$, $\underline{\text{mod}}\, A$: dualizing variety

$$\text{mod}(\text{proj}A) \simeq \text{mod}\, A$$

$\mathcal{C}$: dualizing variety $\Rightarrow$ mod $\mathcal{C}$: abelian

Proposition [K]

$\mathcal{C}$: dualizing variety

- $\widehat{\mathcal{C}}$: dualizing variety
- mod $\widehat{\mathcal{C}}$: abelian Frobenius category

Dualizing variety

Theorem B [K]

$\mathcal{C}$: dualizing variety s.t.

$\operatorname{proj.dim}(M) < \infty \quad \forall M \in \mathsf{mod}\,\mathcal{C}$ and

$\operatorname{proj.dim}(N) < \infty \quad \forall N \in \mathsf{mod}\,\mathcal{C}^{\mathrm{op}}$.

Then we have a triangle equivalence

$$\underline{\mathsf{mod}}\,\widehat{\mathcal{C}} \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\mathcal{C}).$$

This recovers Happel's triangle equivalence

$$\underline{\mathsf{mod}}\,\widehat{A} \simeq \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A),$$

for an algebra A with $\operatorname{gl.dim}(\mathsf{mod}\,A) < \infty$, $\mathcal{C} = \mathsf{proj}A$.

Dualizing variety

Proposition [AR]

A : algebra, $\operatorname{gl.dim}(\operatorname{mod} A) \leq n$
$\Rightarrow \operatorname{gl.dim}(\operatorname{mod}(\underline{\operatorname{mod}} A)) \leq n+1$

For $\mathcal{C} = \underline{\operatorname{mod}} A$, we have

Corollary [K]

A : algebra, $\operatorname{gl.dim}(\operatorname{mod} A) < \infty$ then

$$\underline{\operatorname{mod}}(\widehat{\underline{\operatorname{mod}} A}) \simeq \mathrm{D}^{\mathrm{b}}(\operatorname{mod}(\underline{\operatorname{mod}} A))$$

Main theorem

Theorem [IO, K]

A : hereditary algebra, then

$$\underline{\mathsf{mod}}\,\mathrm{D}^{\mathrm{b}}(\mathsf{mod}\,A) \;\simeq\; \underline{\mathsf{mod}}(\widehat{\underline{\mathsf{mod}}\,A}) \;\simeq\; \mathrm{D}^{\mathrm{b}}(\mathsf{mod}(\underline{\mathsf{mod}}\,A)).$$