On S-Noetherian rings

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S-Noetherian ring and S-Noetherian module

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Definition (2002, Anderson-Dumitrescu)

Let R be a commutative ring with identity, S a multiplicative subset of R and M an R-module.

(1) An ideal I of R is *S-finite* if there exist an $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$.

(2) R is an *S-Noetherian ring* if each ideal of R is S-finite.

(3) M is *S-finite* if there exist an $s \in S$ and a finitely generated R-submodule F of M such that $sM \subseteq F$.

(4) M is *S-Noetherian* if each submodule of M is S-finite.

- If S consists of units of R, then the notion of S-Noetherian rings (resp., S-Noetherian modules) is precisely the same as that of Noetherian rings (resp., Noetherian modules).

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Almost finitely generatedness

Definition (1995, Anderson-Kwak-Zafrullah)

Let D be an integral domain with quotient field K and I a nonzero ideal of $D[X]$.

(1) I is *almost finitely generated* if there exist $f_1, \ldots, f_m \in I$ with $\deg f_i > 0$ and $s \in D \setminus \{0\}$ such that $sI \subseteq (f_1, \ldots, f_m)$.

(2) $D[X]$ is *almost Noetherian* if each nonzero ideal I of $D[X]$ with $IK[X] \neq K[X]$ is almost finitely generated.

(3) D is *agreeable* if for each fractional ideal F of $D[X]$ with $F \subseteq K[X]$, there exists an $s \in D \setminus \{0\}$ with $sF \subseteq D[X]$.

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Basic properties

Proposition (2002, Anderson-Dumitrescu)

Let R be a commutative ring, S a multiplicative subset of R and M an R-module. Then the following assertions hold.

(1) R is S-Noetherian if and only if every prime ideal of R (disjoint from S) is S-finite.

(2) If R is an S-Noetherian ring and M is an S-finite R-module, then M is an S-Noetherian R-module.

(3) If T is both an S-Noetherian ring containing R and an S-finite R-module, then R is an S-Noetherian ring.

Proposition (2002, Anderson-Dumitrescu)

Let $R \subseteq T$ be a ring extension such that $IT \cap R = I$ for each ideal I of R and S a multiplicative subset of R. If T is S-Noetherian, then so is R.

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Amalgamation

Definition

Let $f : A \to B$ be a ring homomorphism and J an ideal of B. The subring $A \bowtie^f J$ of $A \times B$ is defined as follows:

$$A \bowtie^f J = \{(a, f(a) + j) \mid a \in A \text{ and } j \in J\}.$$

We call the ring $A \bowtie^f J$ the *amalgamation of A with B along J with respect to f*.

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Amalgamation (Continued)

In fact, $A \bowtie^f J$ is the pullback $\widehat{f} \times_{B/J} \pi$ of $\widehat{f}$ and π, where $\pi : B \to B/J$ is the canonical projection and $\widehat{f} = \pi \circ f$:

$$\begin{array}{ccc} A \bowtie^f J = \widehat{f} \times_{B/J} \pi & \xrightarrow{p_A} & A \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle \widehat{f}} \\ B & \xrightarrow{\pi} & B/J. \end{array}$$

Also, the map $\imath : A \to A \bowtie^f J$ given by $a \longmapsto (a, f(a))$ for all $a \in A$ is the natural embedding.

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Main result

For a multiplicative subset S of A, put $S' := \{(s, f(s)) \mid s \in S\}$. Clearly, S' and $f(S)$ are multiplicative subsets of $A \bowtie^f J$ and B, respectively.

Theorem

Let $f : A \to B$ be a ring homomorphism, J an ideal of B, S a multiplicative subset of A and $S' := \{(s, f(s)) \mid s \in S\}$.

(1) If A is an S-Noetherian ring and B is an S-finite A-module (with the A-module structure induced by f), then $A \bowtie^f J$ is an S'-Noetherian ring.

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Application 1: Composite ring extensions

Basic setting

- $D \subseteq E$: an extension of commutative rings with identity;
- $\{X_1, \ldots, X_n\}$: a set of indeterminates over E;
- $D + (X_1, \ldots, X_n)E[X_1, \ldots, X_n] := \{f \in E[X_1, \ldots, X_n] \mid$ the constant term of f belongs to $D\}$; and
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Application 1: Composite ring extensions (Continued)

Recall that a multiplicative subset S of a commutative ring R is *anti-Archimedean* if $\bigcap_{n \in \mathbb{N}} s^n R \cap S \neq \emptyset$ for every $s \in S$.

Corollary

Let $D \subseteq E$ be an extension of commutative rings, $\{X_1, \ldots, X_n\}$ a set of indeterminates over E, J an ideal of $E[X_1, \ldots, X_n]$ and S an anti-Archimedean subset of D.

(1) If D is an S-Noetherian ring and E is an S-finite D-module, then $D[X_1, \ldots, X_n] + J$ is an S-Noetherian ring.

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Application 2: Nagata's idealization

Let R be a commutative ring with identity and M a unitary R-module. The *idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) \mid r \in R \text{ and } m \in M\}$$

under the usual addition and the multiplication defined as $(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$.

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Let R be a commutative ring with identity and M a unitary R-module. The *idealization* of M in R (or *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) \mid r \in R \text{ and } m \in M\}$$

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Application 2: Nagata's idealization (Continued)

Basic properties

Let R be a commutative ring with identity and M a unitary R-module. If S is a multiplicative subset of R, then $S(+)M$ is a multiplicative subset of $R(+)M$.

Theorem

Let R be a commutative ring with identity, M a unitary R-module and S a multiplicative subset of R. Then the following statements are equivalent.

(1) $R(+)M$ is an $S(+)M$-Noetherian ring.

(2) R is an S-Noetherian ring and M is S-finite.

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Thank you for your attention!