On Nakayama Conjecture and related conjectures-Review

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温故知新

Review the old knowledge and learn new idea.

環論シンポジュームの曙 (Pioneer of Ring Symposium)

第 6 回代数学シンポジューム (1964 年 7 月 10 日- 14 日)
北海道大学理学部（世話人：東屋五郎）
講演題目

1. 森田氏の定理をめぐって (p.1-7)（東屋五郎：北大）
2. Separable algebra の Galois の理論 (神崎熙夫：大阪学芸大)
3. QF-3 algebra の dominant dimension (太刀川弘幸：京都工芸繊維大)
4. 射影的加群 (I 遠藤静夫：慶応大、 II 日野原幸利：熊本大)
5. フロベニュース拡大 (I 都築俊郎：名古屋大、II 小野寺毅：北大）
6. 可環環上の半単純多元環 (服部昭：東京教育大)
7. Maximal order のホモロジー的考察 (原田学：大阪市大)
8. Profinite group のコホモロジー論と整数論への応用
   (I 河田敬義：東大、II 佐々木良雄：愛媛大）
9. Grothendieck cohomology の紹介 (山田浩：東京教育大)
10. Chen classes と projective class group (尾関英樹：名古屋大)
11. Derive category の理論の紹介 (pp.68-85) (松村英之：京大)

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Nakayama Conjecture

Let A be a finite dimensional algebra over a field K and $D(M) = \mathrm{Hom}_k(M, K)$ a dual space of a vector space M.

Tadashi Nakayama gave the following conjecture in 1958.

Conjecture (NC:Nakayama Conjecture)

Assume ${}_AA$ has a minimal injective resolution

$$0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$$

with all E_i's are projective, then A is self-injective.

Reference: Tadashi Nakayama
On algebras with complete homology,
Abh. Math. Sem. Univ. Hamburg 22 (1958), 300-307.

Tachikawa Conjecture

Hiroyuki Tachikawa gave the following conjecture which is equivalent to NC.

Conjecture (TC: Tachikawa Conjecture)

[T1] $\mathrm{Ext}^i_A({}_A D(A), {}_A A) = 0$ *for all* $i > 0$, *then* A *is self-injective.*

[T2] *Assume* A *is a self-injective algebra and* M *is a finitely generated left* A *module. If* $\mathrm{Ext}^i_A(M, M) = 0$ *for all* $i > 0$, *then* M *is projective.*

Reference: Hiroyuki Tachikawa
Quasi-Frobenius Rings and Generalizayuions, QF-3 and QF-1 Rings
Lecture Notes in Mathematics, Springer-Verlag, Inc., Berlin and New York, 1973

Tachikawa Conjecture

Remark 1.1

[T2] and hence [NC] are not true for an artinian ring in general.
We see this in Chapter 7(7)

[NC] is a typical conjecture for algebras.

What is the difference between algebras and artinian rings ?

New Nakayama Conjecture

In general, an artinian ring has not self-duality, so we give the following new conjecture.

Conjecture (NNC: New Nakayama Conjecture)

Assume an artinian ring A has a self-duality and $_AA$ has a minimal injective resolution

$$0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$$

with all E_i's are projective, then A is self-injective.

Artinian ring with self-duality

Typical example of an artinian ring with self-duality is an artin algebra, which is an artinian ring finitely generated over its center.

An artin algebra was orginally defined by Emil Artin.

Reference: Maurice Auslander, Idun Reiten, Sverre O. Smalo, (1997)[1995], Representation theory of Artin algebras,
Cambridge Studies in Advanced Mathematics, 36,
Cambridge University Press, ISBN $978-0-521-59923-8$,
MR 1314422, Zbl 0834.16001

Ring with self-duality

Yoshitomo Baba's comment for Rings with self-duality

Reference:新しいアルティン環の流れ, 数学 67(3) 2015 年, 271-290 ページ

The following rings are typical rings with self-duality.

(1) commuative ring

(2) Serial ring (Amdal, Ringdal,1968)
 Reference: Catégories uniséraleles, C.R. Acad. Sci. Paris Sér. A*cdot*B, 267 (1968),A85-A87, A247-249.

(3) Harada(H) ring with homogenious socle
 i.e. $\mathrm{soc} R$ is a finite direct sum of a simple module.

(4) Homogenius type Harada ring (Kado and Oshiro, 1999)
 Reference: Self-Duality and Harada Rings, J.Alg. 211,1999,384-408.

A ring R is called left H-ring if for any indecomposable projective right module P_R, there is some indecomposable projective injective right module I such that $P = I\mathrm{rad}^n R$ for some $n > 0$.

Quasi-Harada ring

(5) Some Quasi-Harada(QH) rings

A ring R is called QH-ring if any projective left (right) module is quasi-injective.

Example of a ring of the theorem

Let D be a division ring and set $R = D \times D \times D$ with the multiplication;

$$(x_1, x_2, x_3)(y_1, y_2, y_3) = (x_1y_1, x_1y_2 + x_2y_1, x_1y_3 + x_2y_2 + x_3y_1).$$

Then R is non-commutative local serial ring with loewy length 3 and $(0, 1, 0)$ is in (center of eRe)$\cap(e(\mathrm{rad}A)e - (e(\mathrm{rad}A)e)^2)$.

Theorem 1.2

QH ring is QF-3 ring. (i.e.) There is an idempotent $e \in R$ such that eR is minimal faithful module.

If eRe is local serial and (center of eRe)$\cap(e\mathrm{rad}Ae - (e\mathrm{rad}Ae)^2)$ is not empty, then R has self-duality

The equivalence of [NC] and [TC]

To show the equivalence of [NC] and [TC],
it requires the following facts and notations.

Lemma 1.3

It holds for finite dimensional algebras over a field K

$$\mathrm{Ext}^i_A({}_AD(A), {}_AA) \cong \mathrm{Ext}^i_{A^e}(A, A^e).$$

Here, $A^e = A \otimes_K A^{op}$ is an enveloping algebra of A.

Proof.

$$\begin{aligned}\mathrm{Ext}^i_A(D(A_A), {}_AA) &= \mathrm{Ext}^i_{A\otimes_K K}({}_AA \otimes_A D(A_A)_K, {}_AA_K) \\ &\cong \mathrm{Ext}^i_{A^e}({}_AA_A, \mathrm{Hom}_K(D(A_A)_K, {}_AA_K)).\end{aligned}$$

Also,

$$\begin{aligned}{}_A\mathrm{Hom}_K(D(A_A)_K, {}_AA_K)_A &= {}_A\mathrm{Hom}_K(D(A_A)_K, D({}_KD({}_AA_K)))_A \\ &\cong {}_A\mathrm{Hom}_K(D(A_A) \otimes_K D({}_AA_K), {}_KK)_A \\ &\cong D(D({}_AA \otimes_K A_A)) \\ &\cong {}_AA \otimes_K A_A.\end{aligned}$$

The equivalence of [NC] and [TC]

Definition 1

1. (left dominant dimension)
   We denote $\ell.dom.dimA \geq n$ when A has a minimal injective resolution

   $$0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$$

   with projective modules $E_1, \ldots, E_n$.
2. (left QF-3 ring)
   A is called left QF-3 if it satisfies the one of the following equivalent conditions;
   1. $E(A) \subset \prod A$.
      Here, $E(A)$ is an injective envelop of ${}_AA$.
   2. A has a mininaml faithful module ${}_AM$.
      (i.e.) ${}_AM$ is faithful and for any faithful module ${}_AN$, it holds $N\oplus > M$.
   3. There is an idempotent $f = f^2 \in A$ such that Af is faithful injective.

A is called QF-3 if A is left and right QF-3.

The equivalence of [NC] and [TC]

Lemma 1.4 (LNM351, p.p.97)

Let A be a QF-3 ring with minimal faithful modules Ae and fA. Assume $\ell.\mathrm{dom.dim}\, A \geq 2$ and the first n images of the minimal injective resolution of ${}_{fAf}fA$ are finitely cogenerated by ${}_{fAf}fAe$, then the the following conditions are equivalent.

1. $\ell.\mathrm{dom.dim}\, A \geq n+2$.
2. $\mathrm{Ext}^i_{fAf}(fA, fA) = 0$ *for* $i = 1, 2, \ldots, n$.
3. $\mathrm{Ext}^i_{eAe}(Ae, Ae) = 0$ *for* $i = 1, 2, \ldots, n$.

The equivalence of [NC] and [TC]

Theorem 1.5

$[\mathrm{NC}] \iff [\mathrm{TC}]$

Proof.

Assume [NC]. We first prove [T1].
We set $R = \mathrm{End}_A(A \oplus D(A))$ and f and e projections to A and $D(A)$, respectively.
Then it holds

$fRf = A,\ fR = fRf \oplus fRe = A \oplus D(A)$

as left A-module. Since

$$\begin{aligned} \mathrm{Ext}^i_{fRf}(fR, fR) &= \mathrm{Ext}^i_A(A \oplus D(A), A \oplus D(A)) \\ &= \mathrm{Ext}^i_A(D(A), A), \end{aligned}$$

we have $\mathrm{Ext}^i_{fRf}(fR, fR) = 0$ from [T1].
From Lemma 1.4, we know $\ell.\mathrm{dom.dim}.R = \infty$.
So R is self-injective by [NC].
Thus A is also self-injective.(See Lemma 1.6 below .) □

The equivalence of [NC] and [TC]

Proof.

(Continuous)
Next we prove [T2]. Assume A is self-injective and M is finitely generated.
We set $R = \mathrm{End}_A(A \oplus M)$ and f and e projections to A and M, respectively.
By the same argument in the proof of [T1], it holds
$\mathrm{Ext}^i_{fRf}(fR, fR) = \mathrm{Ext}^i_A(M, M) = 0$ and R is self-injective.

On the other hand, since $A \oplus M$ is finitely generated generator (co-generator), it is well known that this satisfies double centralizer property. i.e.
$\mathrm{End}_R(A \oplus M) = A$.
Hence $A \oplus M$ is a projectice A-module. (See Lemma 1.6.)
Thus M is a projective

□

The equivalence of [NC] and [TC]

Lemma 1.6

(1) *Assume ${}_AM$ is finitely generated and $\mathrm{Ext}^1_A(M, M) = 0$.*
If $R = \mathrm{End}_A(M)$ is right self-injective, then M is a projective $\mathrm{End}_R(M_R)$-module.

(2) *Assume $R = \mathrm{End}_A(A \oplus D(A))$ is self-injective, then $\mathrm{Ext}^1_A(D(A), A) = 0$ iff A is self-injective.*

The equivalence of [NC] and [TC]

Proof.

(1) We take a short exact sequence of left A-modules;

$$0 \to N \to \oplus A \to M \to 0.$$

We apply $\mathrm{Hom}_A(-, M)$ to the above exact sequence,
we have the split short exact sequence of right R-modules

$$0 \leftarrow \mathrm{Hom}_A(N, M) \leftarrow \mathrm{Hom}_A(\oplus A, M) \leftarrow \mathrm{Hom}_A(M, M) = R \leftarrow 0$$

from the assumptions $\mathrm{Ext}^1_A(M, M) = 0$ and R is right self-injective.
We apply $\mathrm{Hom}_R(-, M_R)$ to the above exact sequence, we have the split exact sequence;

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_A(M, N), M) \to \oplus \mathrm{End}_R(M) \to \mathrm{Hom}_R(R, M) = M \to 0.$$

Thus M is a projective $\mathrm{End}_R(M)$-module.

(2) If part is clear, so we prove only if part. We remark $A = \mathrm{End}_R(M)$ since ${}_AM$ is generator. We apply (1) to $M = A \oplus D(A)$, then ${}_AD(A)$ is projective, that is, A_A is injective. So A is self-injective. □

The equivalence of [NC] and [TC]

Lemma 1.7

Let ${}_AM$ be an A-module, $B = \mathrm{End}_A M$ and

$$d: A \to \mathrm{End}_B M_B$$

a canonical map defined by $d(a)(m) = am$ for $a \in A$, and $m \in M$.

(1) *d is monomorphism iff ${}_AM$ is faithful.*

(2) *If ${}_AM$ is generator, then d is an isomorphism and M_B is finitely generated projective.*

(3) *If ${}_AM$ is finitely generated projective, then M_B is finitely generated generator.*

The equivalence of [NC] and [TC]

Proof.

(1) is clear.
(2) Since generator is faithful, so d is monomorphism.
So we show d is an epimorphism.
Take an epimorphism $\sum^{n} \oplus M \xrightarrow{(f_1, f_2, \cdots, f_n)} A$,
then there are some $m_i \in M$ ($j = 1, \cdots, n$) such that

$$1_A = f_1(m_1) + f_2(m_2) + \cdots + f_n(m_n).$$

Also for $m \in M$, we define $\phi_m : {}_AA \to {}_AM$ by $\phi_m(a) = am$ for any $a \in A$.
We remark $f_j\phi_m \in B$. For any $\varphi \in \mathrm{End}_B(M_B)$,

$$\varphi(m_j \cdot f_j\phi_{m_i}) = \varphi(m_j) \cdot f_j\phi_{m_i} = f_j(\varphi(m_j))m_i \in Am_i.$$

Since

$$\sum_{j=1}^{n} f_j(m_j)m_i = (\sum_{j=1}^{n} f_j(m_j))m_i = m_i,$$

we have

$$\varphi(m_i) = (\sum_{j=1}^{n} f_j(\varphi(m_j)))m_i \in Am_i.$$

The equivalence of [NC] and [TC]

Proof.

We set $\varphi(m_i) = a_i m_i$ and $a = a_1 f_1(m_1) + a_2 f_2(m_2) + \cdots + a_n f_n(m_n)$, then for any $m \in M$,

$$\begin{aligned} m = 1 \cdot m &= f_1(m_1)m + f_2(m_2)m + \cdots + f_n(m_n)m \\ &= m_1(f_1\varphi_m) + \cdots + m_n(f_n\varphi_m) \end{aligned}$$

So

$$\begin{aligned} \varphi(m) &= \varphi(m_1)f_1\varphi_m + \cdots + \varphi(m_n)f_n\varphi_m \\ &= (a_1 f(m_1) + \cdots + a_n f(m_n))m \\ &= am \end{aligned}$$

We apply $\mathrm{Hom}_A(-, {}_AM_B)$ to the above a splittable epimorphism, then we have a splittable epimorphism

$$\sum^n \oplus \mathrm{Hom}_A(M, {}_AM_B)_B = \sum^n \oplus B_B \to \mathrm{Hom}_A(A, {}_AM_B)_B = M_B \to 0.$$

Thus M_B is finitely generated projective. □

The equivalence of [NC] and [TC]

Proof.

(3) Assume ${}_AM$ is finitely generated projective, then we have a splittable epimorphim

$$\sum^n \oplus {}_AA \xrightarrow{(f_1,f_2,\cdots,f_n)} {}_AM \to 0.$$

That is, there are $f_i(1) = m_i \in M$ and $g_i : {}_AM \to {}_AA$ $(i = 1, \cdots, n)$ such that

$$m = m_1g_1(m) + m_2g_2(m) + \cdots + m_ng_n(m)$$

for any m. Hence $m = m_1(g_1\varphi_m) + \cdots + m_n(g_n\varphi_m)$.
Remarking that $g_i\varphi_i \in B$, $m_1, \cdots, m_n$ are generators of M_B, that is, M_B is finitely generated B-module.
Apply $\mathrm{Hom}_A(-, {}_AM_B)$ to the above splittable exact sequence, we have a splittable epimorphism

$$\sum^n \oplus \mathrm{Hom}_A(A, {}_AM_B) = \sum^n \oplus M_B \to \mathrm{End}_A(M) = B_B \to 0.$$

That is, M_B is generator. □

Tachikawa Conjecture +

In the proof of [NC]$\Longleftrightarrow$ [TC],
the properties of generator and co-generator are essential.
So Tachikawa gave the following conjecture equivalent to [TC] by using the notion of generator and co-generator.

Conjecture (TC+: Tachikawa Conjecture +)

Let ${}_AM$ be finitely generated generator co-generator.
If $\mathrm{Ext}^i_A(M, M) = 0$ *for any* $i > 0$, *then* M *is projective.*

Theorem 2.1

[TC] $\Longleftrightarrow$ [TC+]

Tachikawa Conjecture +

Proof.

Assume [TC].
Since M is generator co-generator, we have a splittable epimorphism $\sum \oplus M \to A \to 0$.
That is, for some $m, n > 0$, it holds ${}_A A < \oplus M^{(n)}$ and ${}_A D(A) < \oplus M^{(m)}$.
Thus $\mathrm{Ext}^i_A(M, M) = 0$ implies $\mathrm{Ext}^i_A(D(A), A) = 0$.
[T1] implies A is self-injective. Hence M is projective by [T2].

□

Tachikawa Conjecture +

Proof.

Assume [TC+], then we have

$$0 = \mathrm{Ext}^i_A(D(A), A) = \mathrm{Ext}^i_A(D(A) \oplus A, D(A) \oplus A).$$

We show [T1]. Since $D(A) \oplus A$ is projective., $A = D(D(A))$ is injective.
We show [T2] $\mathrm{Ext}^i_A(M, M) = 0$ for $i > 0$ and A is self-injective implies $\mathrm{Ext}^i_A(M \oplus A, M \oplus A) = 0$,
Also $D(A) = A$ implies $D(A)$ is co-generator,
thus $M \oplus D(A)$ is finitely generated generator cogenrator.
By [TC+], ${}_AM$ is projective. □

Generalized Nakayama Conjecture

Mauris Auslnder and Idun Reiten gave the following conjecture in 1975.

Conjecture (GNC: Generalized Nakayama Conjecture)

Let $0 \to A \to E_1 \to E_2 \to \cdots \to E_n \to \cdots$ *be a minimal injective resolution of* ${}_AA$ *and* S *any simple module, then there is some* i *such that* $S < E_i$

Reference:Maurice Auslander and Idun Reiten,
On a generalized version of the Nakayama conjecture,
Proc. Amer. Math. Soc. 52 (1975), 69-74.

Remark 3.1

$[\mathrm{GNC}] \iff \mathrm{Ext}^i_A(S, A) \neq 0$ for some $i > 0$

Generalized Nakayama Conjecture+

Conjecture (GNC+: Generalized Nakayama Conjecture+)

A generator ${}_AM$ satisfying $\mathrm{Ext}^i_A(M, M) = 0$ for any $i > 0$ is finitely generated projective.

Theorem 3.2

$[\mathrm{GNC}] \iff [\mathrm{GNC+}]$

Particularly $[\mathrm{GNC}] \implies [\mathrm{NC}]$

Generalized Nakayama Conjecture+

Proof.

Assume [GNC].
We set $B = \mathrm{End}_A(M)$. Then M_B is finitely generated projective since ${}_AM$ is generator .
Let

$$0 \to {}_AM \to E_1 \to E_2 \cdots$$

be a minimal injective resolution of ${}_AM$.
We apply $\mathrm{Hom}_A(M, -)$, then the following sequence

$$\begin{aligned} 0 \to B = {}_B\mathrm{Hom}_A({}_AM_B, {}_AM) \\ \to {}_B\mathrm{Hom}_A({}_AM_B, E_1) \to {}_B\mathrm{Hom}_A({}_AM_B, E_2) \to \cdots \end{aligned}$$

is exact since $\mathrm{Ext}^i_A(M, M) = 0$ for any $i > 0$.
Also ${}_B\mathrm{Hom}_A({}_AM_B, {}_AE_i)$ is injective since M_B is projective and ${}_AE_i$ is injective.
Thus for some $m >> 0$, $\sum_{i=1}^{m} \oplus \mathrm{Hom}A({}_AM_B, {}_AE_i)$ is co-generator by [GNC]. □

Generalized Nakayama Conjecture+

Proof.

On the other hand,

${}_AE_i < \oplus \sum^{t_i} \oplus D(A)$ since $D(A)$ is an injective co-generator.

So ${}_B\mathrm{Hom}_A({}_AM_B, E_i) < \oplus \sum^{t_i} \oplus {}_B\mathrm{Hom}_A({}_AM_B, D(A))$.
Since ${}_B\mathrm{Hom}_A({}_AM_B, D(A)) \cong {}_B\mathrm{Hom}_A(A \otimes_A M_B, A) = D(M_B)$ and $D(M_B)$ is co-generator, so M_B is generator.

Thus ${}_AM$ is finitely generated projective.Hence [GNC+] holds. □

Generalized Nakayama Conjecture+

Proof.

We assume [GNC+].
Let

$$0 \to {}_A A \to E_1 \to E_2 \to \cdots$$

be a minimal injective resolution of ${}_A A$ and $\{S_1, S_2, \dots S_n\}$ the complete set of non-isomorphic simple modules included in some E_i.
We take $f \in A$ such that $f^2 = f$ and

$${}_A E(S_1) \oplus {}_A E(S_2) \oplus \cdots {}_A E(S_n) = {}_A D(fA).$$

Thus there is some m_i such that $E_i < \oplus_A D(fA)^{m_i}$.
Remarking that $fA \otimes_A D(fA) = fD(fA) = D(fAf)$ as left as fAf-module, we have natural isomorphisms

$$\begin{aligned} {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A D(fA)) &\cong {}_A\mathrm{Hom}_{fAf}(fA, D(fAf)) \\ &\cong {}_A\mathrm{Hom}_K(fAf \otimes_{fAf} fA_A, K) \\ &= {}_A D(fA_A). \end{aligned}$$

□

Generalized Nakayama Conjecture+

Proof.

Hence we have natural isomorphism

$$\varphi_i : \ {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes E_i) \cong {}_A E_i.$$

Make an exact commutative diagram form an exact sequence $0 \to {}_A A \to E_1 \to E_2$,

$$\begin{array}{ccccccc}
0 \longrightarrow & {}_A A & \longrightarrow & E_1 & \longrightarrow & E_2 \\
 & & & \downarrow \varphi_1 & & \downarrow \varphi_2 \\
0 \longrightarrow & {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A A) & \longrightarrow & {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A E1) & \longrightarrow & {}_A\mathrm{Hom}_{fAf}(fA, fA \otimes_A E2).
\end{array}$$

Thus ${}_A A \cong \mathrm{End}_{fAf}(fA)$.

On the other hand, $fA \otimes_A D(fA) = D(fAf)$ is ab injective fAf-module, so is $fA \otimes_A E_i$.

Hence we have an injective resolution of ${}_{fAf} fA = {}_{fAf} fA \otimes_A A$

$$0 \to fA \otimes_A A \to fA \otimes_A E_1 \to fA \otimes_A E_2 \to \cdots .$$

Generalized Nakayama Conjecture+

Proof.

From the above two facts, we have $\mathrm{Ext}^i_{fAf}(fA, fA) = 0$.
Hence ${}_{fAf}fA$ is finitely generated projective by [GNC+],
so fA is a generator as left $\mathrm{End}_{fAf}(fA)(= A)$-module,
that is, fA_A is a finitely generated projective generator.
Thus ${}_AD(fA)$ is co-generator, which means $\{S_1, S_2, \dots S_n\}$ is the complete set of all non-isomorphic simple modules.
Hence [GNC] holds. □

Strong Nakayama Conjecture

Robert R. Colby and Kent R. Fuller gave the following conjecture in 1990.

Conjecture (SNC: Strong Nakayama Conjecture)

For any finitely generated module ${}_AM$, there is some $i \geq 0$ such that $\mathrm{Ext}^i_A(M, A) \neq 0$.

Reference: Robert R. Colby and Kent R. Fuller,
A NOTE ON THE NAKAYAMA CONJECTURES,
TSUKUBA J. MATH. Vol. 14 No. 2 (1990), 343—352

Remark 4.1

$[\mathrm{SNC}] \implies [\mathrm{GNC}]$ is clear.

Finitistic Dimension Conjecture

The finitistic dimension of an algebra A is defined by

$$\mathrm{f.gl.dim} A = \sup\{\mathrm{p.d}(M) < \infty\}$$

Here, $\mathrm{p.d}(M)$ is a projective dimension of ${}_AM$.

Conjecture (FDC: Finitistic Dimension Conjecture)

$$\mathrm{f.gl.dim} A < \infty$$

Theorem 5.1

$$[\mathrm{FDC}] \implies [\mathrm{SNC}]$$

Finitistic Dimension Conjecture

Proof.

Assume $n = \mathrm{f.gl.dim} A < \infty$.
Take ${}_A M$ such that $\mathrm{Ext}^i_A(M, A) = 0$ for all $i \geq 0$.
Let

$$\cdots \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \xrightarrow{f_0} M \to 0$$

be a projective resolution of ${}_A M$.
Then by assumption, we have an exact sequence

$$\cdots \leftarrow \mathrm{Hom}_A(P_1, A)_A \xleftarrow{\mathrm{Hom}_A(f_1,A)} \mathrm{Hom}_A(P_0, A)_A \xleftarrow{\mathrm{Hom}_A(f_0,A)} \mathrm{Hom}_A(M, A)_A = 0.$$

So we have the projective resolution of $\mathrm{ImHom}_A(f_{n+2}, A)$

$$0 \leftarrow \mathrm{ImHom}_A(f_{n+2}, A) \xleftarrow{\mathrm{Hom}_A(f_{n+2},A)} \mathrm{Hom}_A(P_{n+1}, A)_A \leftarrow \cdots \leftarrow \mathrm{Hom}_A(P_1, A)_A \xleftarrow{\mathrm{Hom}_A(f_1,A)} \mathrm{Hom}_A(P_0, A)_A \xleftarrow{\mathrm{Hom}_A(f_0,A)} \mathrm{Hom}_A(M, A)_A = 0.$$

□

Finitistic Dimension Conjecture

Proof.

Since $\mathrm{p.d}\ \mathrm{Im}\mathrm{Hom}_A(f_{n+2}, A) \leq n$, we have a splittable epimorphism

$$0 \leftarrow \mathrm{Hom}_A(P_1, A)_A \xleftarrow{\mathrm{Hom}_A(f_1, A)} \mathrm{Hom}_A(P_0, A)_A.$$

Thus we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_A(\mathrm{Hom}_A(P_1, A)_A, A_A) & \xrightarrow{g} & \mathrm{Hom}_A(\mathrm{Hom}_A(P_0, A)_A, A_A) & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & \\ P_1 & \xrightarrow{f_1} & P_0 & \longrightarrow & 0. \end{array}$$

Here $g = \mathrm{Hom}_A(\mathrm{Hom}_A(f_1 A_A)$.

Thus f_1 is splittable epimorphism, which means $M = 0$.

□

Tilting version of Generalized Nakayama Conjecture

Takayashi Wakamatsu gave the following conjecture.

Conjecture (TGNC: Tilting version of Generalied Nakayama Conjecture)

Assume T_A is a tilting module and let

$$0 \to A \to T_1 \to T_2 \to \cdots \to T_n \to \cdots$$

be a minimal dominant resolution,
then for any indecomposable direct summand $T' < \oplus T$,
there is some i such that $T' < \oplus T_i$.

A module T_A is called a tilting module if the following two conditions are satisfied;
(1) $\mathrm{Ext}^i_A(T, T) = 0$ for any $i > 0$.
(2) There is some exact sequence

$$\cdots \to T_2 \to T_1 \to D(A)_A \to 0$$

such that $T_i < \oplus(\sum^{n_i} \oplus T)$ for every i and

$$\mathrm{Hom}_A(T, T_2) \to \mathrm{Hom}_A(T, T_1) \to \mathrm{Hom}_A(T, D(A)) \to 0.$$

Tilting version of Generalized Nakayama Conjecture

Takayoshi Wakamatsu gave the following conjecture in his lecture which is equivalent to [GNC].

Theorem 6.1

[GNC] $\Longleftrightarrow$ [TGNC]

Let T_A be a tilting module. An exact sequence

$$0 \to A_A \to T_1 \to T_2 \to \cdots$$

is called a dominant resolution if the following two conditions are satisfied;

(1) $T_i < \oplus(\sum^{n_i} \oplus T_A)$ for every i.

(2) $0 \leftarrow \mathrm{Hom}_A(A, T) \leftarrow \mathrm{Hom}_A(T_1, T) \leftarrow \mathrm{Hom}_A(T_2, T) \leftarrow \cdots$ is exact.

Remark 6.2 (Wakamatsu)

There is a minimal dominant resolution.

Tilting version of Generalized Nakayama Conjecture

Proof.

Assume [TGNC]. Let (*) $0 \to {}_AA \to {}_AI_1 \to {}_AI_2 \to \cdots$ be a minimal injective resolution of ${}_AA$. ${}_AD(A_A)$ is a tilting module with a minimal dominant resolution (*). Indecomposable direct summands of ${}_AD(A_A)$ are injective envelops of all simple modules. That is, any simple module is a submodule of some I_i by [TGNC]. Hence [GNC] holds. □

Tilting version of Generalized Nakayama Conjecture

Proof.

Next assume [GNC]. We set $B = \mathrm{End}_A(T_A)$. We know that a tilting module has the double centralizer property, we have $A = \mathrm{End}_B({}_BT)$.
Let $0 \to A_A \to T_1 \to T_2 \to \cdots$ and $0 \to {}_BB \to T'_1 \to T'_2 \to \cdots$ be minimal dominant resolutions of A_A and ${}_BB$, respectively.
We take a direct sum $\sum \oplus L$ of non-isomorphic indecomposable direct summands of some T_i.
Since $\sum \oplus L < \oplus T$, there is $f \in B$ such that $f^2 = f$ and $\sum \oplus L = fT$.
[TGNC] is equivalent to $f = 1_B$, so we show $f = 1_B$. □

Tilting version of Generalized Nakayama Conjecture

Proof.

Also we take a direct sum $\sum \oplus M$ of non-isomorphic indecomposable direct summands of some T'_i.
By the same argument as above, there is $e \in A = \mathrm{End}(T_B)$ such that $e^2 = e$ and $\sum \oplus M = Te$.
We know that
(1) ${}_{fAf}fTe_{eAe}$, ${}_B Bf_{fBf}$, ${}_{eAe}eA_A$ are tilting modules.
(2) ${}_B T_A \cong {}_B Bf \otimes_{fBf} fTe \otimes_{eAe} eA$.
(3) ${}_B T_A$ is a tilting module iff ${}_A\mathrm{Hom}_K(T, K)_B = D(T)$ is a co-tilting module.
Since ${}_B Bf_{fBf}$ is a tilting module, there is an exact sequence

$$\cdots \to \sum \oplus Bf \to \sum \oplus Bf \to {}_B D(B) \to 0.$$

Hence we have an exact sequence

$$0 \to B \to \sum \oplus fD(B) \to \sum \oplus fD(B) \to \cdots ,$$

hence $f = 1_B$ by [GNC].

Related Results

Summary:
For algebras,

$[\text{FDC}] \Longrightarrow [\text{SNC}] \Longrightarrow [\text{GNC}] \Longleftrightarrow [\text{GNC+}] \Longleftrightarrow [\text{TGNC}]$

$\Longrightarrow [\text{NC}] \Longleftrightarrow [\text{TC+}] \Longleftrightarrow [\text{TC}] \Longleftrightarrow [\text{TC1}]$ and $[\text{TC2}]$

[NNC] for atinian rings $\Longrightarrow$ [NC] for algebrs

Related Results

1. GEORGE V. WILSON, The Cartan Map on Categories of Graded Modules JOURNAL OF ALGEBRA 85, 390-398 (1983)

Theorem 7.1

[GNC] *is true for positive graded algebras.*

2. Hiroyuki Tachikawa, LNM351, 1984

Theorem 7.2

[T2] *is true for a group algebra* $k[G]$ *for a finite p-group* G *and a field* k.

3. RAINER SCHULTZ, Boundedness and Periodicity of Modules over QF Rings, JOURNAL OF ALGEBRA 101, 450-469 (1986)

Theorem 7.3

[T2] *is true for a group algebra* $k[G]$ *for a finite group* G *and a field* k.

Related Results

1. Edward L. Green, Birge Zimmermann-Huisgen
   Finitistic dimension of artinian rings with vanishing radical cube,
   Mathematische Zeitschrift 206, 505-526 (1991)

Theorem 7.4

[FDC] *is true for an algebra A with vanishing radical cube* (i.e. $\mathrm{rad}^3 A = 0$).

2. Peter Dräxler, A proof of the generalized Nakayama conjecture for algebras with $J^{2\ell+1} = 0$ and A/J^{ℓ} representation finite,
   Journal of Pure and Applied Algebra 78(2), 161-164 (1992)

Theorem 7.5

[GNC] *is true for algebras A with $\mathrm{rad}^{2\ell+1} A = 0$ and $A/\mathrm{rad}^{\ell} A$ representation finite.*

Yong Wang's Result

1. Yong Wang, A remarks on the Strong Nakayama Conjecture,1992

Theorem 7.6

[SNC] *is true for artinian rings* R *with* $\mathrm{rad}^{2\ell+1}R = 0$ *and* $A/\mathrm{rad}^{\ell}R$ *representation finite.*

His proof is very smart !
He uses the fact that $\mathbb{Z}^{2m}$ is a noetherian $\mathbb{Z}$-module.

Yong Wang's Result

Proof.

(Wang's proof)
Assume there is finitely generated non-zero R-module ${}_R M$ such that $\mathrm{Ext}^i_R(M, R) = 0$ for all $i \geq 0$.
For a projective resolution of M,

$$\cdots \to P_{n+1} \xrightarrow{f_{n+1}} P_n \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0,$$

we set $\Omega_i = \mathrm{Im} f_i$ and denote $T^* = \mathrm{Hom}_R({}_R T, {}_R R_R)_R$.
By assumption, we have an exact sequence

$$0 \leftarrow \Omega_i^* \leftarrow P_{i-1}^* \leftarrow \cdots \leftarrow P_1^* \xleftarrow{f_1^*} P_0^* \leftarrow 0..$$

Thus $\mathrm{p.d}\, \Omega_i^* \leq i-1$ for any $i \geq 1$.
Since $\Omega_i^* \subset JP_i^*$, $J^{2\ell}\Omega_i^* = 0$ for any i. □

Yong Wang's Result

Proof.

We prove $\mathrm{Ext}^1_R(\Omega_2^*, R) \neq 0$ and $\mathrm{Ext}^1_R(\Omega_i^*, R) = 0$ for any $i \geq 3$.
Since $P \cong P^{**}$ for any projective module P,
we have the following commutative diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_{n+1} & \xrightarrow{f_{n+1}} & P_n & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{f_1} & P_0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
\cdots & \longrightarrow & P_{n+1}^{**} & \xrightarrow{f_{n+1}^{**}} & P_n^{**} & \longrightarrow & \cdots & \longrightarrow & P_1^{**} & \xrightarrow{f_1{}^{**}} & P_0^{**}.
\end{array}$$

Thus

$$0 \longleftarrow (\Omega_{n+3})^* \longleftarrow (P_{n+2})^* \longleftarrow (P_{n+1})^* \longleftarrow \cdots$$

is a projective resolution and

$$0 \longrightarrow (\Omega_{n+3})^{**} \longrightarrow (P_{n+2})^{**} \longrightarrow (P_{n+1})^{**}$$

Yong Wang's Result

Proof.

Consider the following commutative diagram ;

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & \Omega_2^{**} & \longrightarrow & P_1^{**} & \xrightarrow{f_1^{**}} & P_0^{**} & \longrightarrow & \operatorname{Ext}_R^1(\Omega_2^*, R) & \longrightarrow & 0 \\
 & & & & \cong\downarrow & & \cong\downarrow & & & & \\
 & & & & P_1 & \xrightarrow{f_1} & P_0 & \longrightarrow & M & \longrightarrow & 0,
\end{array}
$$

we know that f_1 is non-splittable.
Thus $\operatorname{Ext}_R^1(\Omega 21^*, R) \neq 0$. □

Yong Wang's Result

Proof.

We fix $m \geq 1$. Take $0 \neq {}_R N$ such that $\mathrm{p.d}\ {}_R N \leq m$, and $J^{2\ell} N = 0$. We set $N_1 = J^\ell N,\ N_2 = N/J^\ell N$.
Since R/J^ℓ is representation finite, let $\{C_1, \ldots, C_m\}$ be the complete set of non-isomorphic indecomposable modules and we have the decompositions

$$N_1 = \sum_{j=1}^{m} \oplus C_j^{a_j},\ N_2 = \sum_{j=1}^{m} \oplus C_j^{b_j}.$$

. For $i > m$, $\mathrm{Ext}_R^{i+1}(N_1, R)_R \cong \mathrm{Ext}_R^i(N_2, R)$ is finitely generated.
We set $\ell(k, j) = \mathrm{length}\ \mathrm{Ext}_R^k(C_j, R)_R$, then $\sum_{j=1}^{m} \ell(i+1, j) \cdot a_j = \sum_{j=1}^{m} \ell(i, j) \cdot b_j$. □

Yong Wang's Result

Proof.

We denote $\mathbb{Z}$-module L_i $(i > 0)$ by

$\{(c_1, \cdots, c_m, d_1, \cdots, d_m) \in \mathbb{Z}^{2m} \mid \sum_{j=1}^{m} \ell(i+1, j) \cdot c_j = \sum_{j=1}^{m} \ell(i, j) \cdot d_j\}$

They are $\mathbb{Z}$-submodules of the noetherian module $\mathbb{Z}^{2m}$.
So an increasing sequence $L_0 \subset L_1 \subset \cdots$ terminates.
That is, $L_{m_0} = L_{m_0+1} = \ldots.$ for some m_0.
Take $N = (\Omega_{m_0+3})^*$. Remarking that $\mathrm{p.d}\ (\Omega_{m_0+3})^* < m_0 + 2$,
$(a_1, \cdots, a_m, b_1, \cdots, b_m) \in L_{m_0+2}$,
thus (*) $(a_1, \cdots, a_m, b_1, \cdots, b_m) \in L_{m_0} = L_{m_0+1}$. □

Yong Wang's Result

Proof.

From the exact sequence

$$0 \to J^\ell N \to N \to N/J^\ell N \to 0$$

and the fact

$$\mathrm{Ext}_R^{m_0+1}((\Omega_{m_0+3})^*, R) \cong \mathrm{Ext}_R^1(\Omega_3^*, R) = 0,$$

we have an exact sequence

$$\begin{aligned} 0 &\to \mathrm{Ext}_R^{m_0+1}(J^\ell N, R) \to \mathrm{Ext}_R^{m_0+2}(N/J^\ell N, R) \to \\ &\quad \mathrm{Ext}_R^{m_0+2}(N, R) \to \mathrm{Ext}_R^{m_0+2}(J^\ell N, R) \to \\ &\quad\quad \mathrm{Ext}_R^{m_0+3}(N/J^\ell N, R) \to \mathrm{Ext}_R^{m_0+3}(N, R) = 0. \end{aligned}$$

□

Yong Wang's Result

Proof.

From (*), we have
length $\mathrm{Ext}_R^{m_0+1}(J^\ell N, R) =$ length $\mathrm{Ext}_R^{m_0+2}(N/J^\ell N, R)$
length $\mathrm{Ext}_R^{m_0+2}(J^\ell N, R) =$ length $\mathrm{Ext}_R^{m_0+3}(N/J^\ell N, R)$
Thus

$$0 = \mathrm{Ext}_R^{m_0+2}((\Omega_{m_0+3})^*, R) = \mathrm{Ext}_R^1(\Omega_2^*, R) \neq 0,$$

which is a contradiction. □

RAINER SCHULTZ's Result

1. RAINER SCHULTZ gave the following example from which we know that **[T2] is not true for artinian rings in general** by Lemma 1.7.

Thus
[NC] is not true for artinian rings in general .

Example 1

There is a self-injective artinian ring R and a finitely generated left R-module ${}_RM$ such that

(i) $\mathrm{Ext}^i_R(M, M) = 0$ for any $i > 0$,
(ii) $M_{\mathrm{End}_R(M,M)}$ is not finitely generated $\mathrm{End}_R(M, M)$-module.

Robert Martinez-Villa's Result

1. Robert Martinez-Villa explored conditions in the category of functors of the stable category which are equivalent to [NC].

Theorem 7.7

Assume $\ell.\mathrm{dom.dim}\ A \leq n$.
Then $\mathrm{Dom}_k = \{{}_AM |\ \ell.\mathrm{dom.dim}\ M \geq k\}$ is contravariantly finite for any $k \leq n$ in the stable category $\underline{\mathrm{mod}}\text{-}A$ of the module category.

We set

$$\tilde{\mathcal{F}}_k = \{F \in \mathrm{mod}(\underline{\mathrm{mod}}\text{-}A) | F(M) = 0 \text{ for any } M \in \mathrm{Dom}_k\}$$

$$\tilde{\mathcal{T}}_k = \{G \in \mathrm{mod}(\underline{\mathrm{mod}}\text{-}A) | G(M) = 0 \text{ for any } M \in \tilde{\mathcal{F}}_k\}$$

Then we know $(\tilde{\mathcal{T}}_k, \tilde{\mathcal{F}}_k)$ is a hereditary torsion theory with a torsion radical t_k.

We denote $\mathrm{Dom} = \bigcap_{k=0}^{\infty} \mathrm{Dom}_k$ and $\tilde{\mathcal{F}} = \bigcap_{k=0}^{\infty} \tilde{\mathcal{F}}_k$.

Let $(\tilde{\mathcal{T}},\ \tilde{\mathcal{F}})$ be a corresponding torsion theory with a torsion radical t.

Robert Martinez-Villa's Result

Robert Martinez-Villa gave the following conjecture.

Conjecture (MC: Martinez Conjecture)

For any $M \in \mathrm{mod}(\underline{\mathrm{mod}}\text{-}A)$, *it holds that*

(1) $t(M) = \bigcap_{k=0}^{\infty} t_k(M)$

(2) $t(M)$ *is finitely presented*

Theorem 7.8

[MC] implies [NC].

Reference: Martinez-Villa, Roberto
Algebras of infinite dominant dimension and torsion theories
Comm. Algebra 22 (1994), no. 11, 4519–4535.

Cheng Chang Xi's Result

1. Cheng Chang Xi showed that
   dominant dimension is not invariant under derived equivalences.

Reference: Cheng Chang Xi
Dominant dimensions, derived equivalences and tilting modules
ISRAEL JOURNAL OF MATHEMATICS 215(2016), no1, 349-395.

Thank you for your attention !