Strongly quasi-hereditary algebras and rejective subcategories

Mayu Tsukamoto

Osaka City University

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Motivation

- Quasi-hereditary algebra
  - heredity chain
  - (co)standard modules (highest weight category)

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Question

Can RSQH be characterised using properties of heredity chains?

Aim

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Characterise **RSQH** by using

- right-strongly heredity chain
- right rejective chain
- coreflective chain

Notation

A : finite dimensional algebra over a field
$\operatorname{mod} A$: the cat. of finitely generated right A-modules
$\operatorname{proj} A$: the full subcat. of $\operatorname{mod} A$ consisting of projective A-modules

Right rejective chains

$\mathcal{C}$: Krull-Schmidt category
$\mathcal{C}'$: full subcat. of $\mathcal{C}$ closed under isom., direct sums and direct summands

Definition [Iyama (2003)]

- $\mathcal{C}'$: **right rejective subcategory** of $\mathcal{C}$ if $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a right adjoint with a counit ε s.t. ε_X is monic for all $X \in \mathcal{C}$.

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- A chain of subcategories of $\mathcal{C}$

$$0 = \mathcal{C}_n \subset \mathcal{C}_{n-1} \subset \cdots \subset \mathcal{C}_i \subset \cdots \subset \mathcal{C}_0 = \mathcal{C}$$

is called a **right rejective chain** if $\mathcal{C}_i$ is a right rejective subcategory of $\mathcal{C}$ and the quotient cat. $\mathcal{C}_i/[\mathcal{C}_{i+1}]$ is semisimple $\forall i$.

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Example

B : factor algebra of A.
mod B is a right rejective subcategory of mod A.

Quasi-hereditary algebras

Assume that A has a chain of two-sided ideals generated by idempotents

$$0 = Ae_nA < \cdots < Ae_{i+1}A < Ae_iA < \cdots < Ae_0A = A \qquad (*)$$

which satisfies $(Ae_iA/Ae_{i+1}A)J(A/Ae_{i+1}A)(Ae_iA/Ae_{i+1}A) = 0$.

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Definition [Cline-Parshall-Scott (1988)]

A : quasi-hereditary (**QH**) $:\Leftrightarrow$ $Ae_iA/Ae_{i+1}A \in \operatorname{proj} A/Ae_{i+1}A$ for $\forall i$.
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Proposition [T (2017)]

A : **RSQH** (resp. **SQH**) $\Leftrightarrow Ae_iA \in \operatorname{proj} A$ (resp. $Ae_iA \in \operatorname{proj} A \cap \operatorname{proj} A^{\mathrm{op}}$).
$(*)$ is called a right-strongly (resp. strongly) heredity chain.

Remark

SQH $\Rightarrow$ RSQH $\Rightarrow$ QH

Main result

Theorem [T (2017)]

The following conditions are equivalent:

1. A : RSQH
   $\Leftrightarrow \exists\ 0 = Ae_nA < \cdots < Ae_iA < \cdots < Ae_0A = A$.
2. $\operatorname{proj} A$ has a right rejective chain
   $$0 = \operatorname{add} e_nA \subset \cdots \subset \operatorname{add} e_iA \subset \cdots \subset \operatorname{add} e_0A = \operatorname{proj} A.$$
3. A has a heredity chain (i.e., A : QH)
   $$0 = Ae_nA < \cdots < Ae_iA < \cdots < Ae_0A = A$$
   and $\operatorname{proj} A$ has a coreflective chain
   $$0 = \operatorname{add} e_nA \subset \cdots \subset \operatorname{add} e_iA \subset \cdots \subset \operatorname{add} e_0A = \operatorname{proj} A.$$

Moreover, A : SQH $\Leftrightarrow$ $\operatorname{proj} A$ has a (right and left) rejective chain.

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Moreover, A : SQH $\Leftrightarrow$ $\operatorname{proj} A$ has a (right and left) rejective chain.

Key Fact [Iyama (2003)]

$AeA \in \operatorname{proj} A \Leftrightarrow \operatorname{add} eA$: right rejective subcat. of $\operatorname{proj} A$.

Application 1

Theorem [T (2017)]

$\mathrm{gldim} A \leq 2 \Rightarrow A :$ RSQH.

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Corollary [Dlab-Ringel (1989)]

$\mathrm{gldim} A \leq 2 \Rightarrow A:$ QH.

Corollary [Ringel (2010)]

Any Auslander algebra is RSQH.

Application 2

A : representation-finite algebra
M : additive generator of $\operatorname{mod} A$
$B := \operatorname{End}_A(M)$: the Auslander algebra of A

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Theorem [Eiriksson (2016)]

B : SQH $\Leftrightarrow$ A : Nakayama algebra

Idea of proof [T (2017)]

- B : SQH $\Leftrightarrow$ proj B has a rejective chain. ($\because$ **Main Theorem**)
- proj $B \simeq$ add $M =$ mod A.
  $\therefore$ B : SQH $\Leftrightarrow$ mod A has a rejective chain.
- $\mathcal{C}' \subset$ mod A : rejective $\Leftrightarrow$ $\exists I$: ideal of A s.t. $\mathcal{C}' =$ mod A/I