Wide subcategories are semistable

based on arXiv:1705.07636

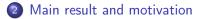
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Today's talk

Wide/Semistable subcategories





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- $\bullet \mbox{ mod } \Lambda$: the category of finitely generated right $\Lambda\mbox{-modules}.$

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A stability condition θ on mod Λ is a linear form on $K_0(\text{mod }\Lambda) \otimes_{\mathbb{Z}} \mathbb{R}$, where $K_0(\text{mod }\Lambda)$ is the Grothendieck group of mod Λ .

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Definition

A semistable subcategory of mod Λ is a full subcategory of θ -semistable Λ -modules for some stability condition θ .

Lemma

Semistable subcategories are wide subcategories.

In general, the converse is not true.

Main result

We proved the following theorem.

Theorem

Left finite wide subcategories of mod Λ are semistable subcategories.

Theorem (Ingalls-Thomas, 2009)

For the path algebra kQ of a finite connected acyclic quiver Q over a field k, there are bijections between the following objects:

- (1) Isomorphism classes of basic support tilting modules in mod(kQ).
- (2) Functorially finite torsion classes in mod(kQ).
- (3) Functorially finite wide subcategories of mod(kQ).
- (4) Functorially finite semistable subcategories of mod(kQ).

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It was generalized for any finite dimensional algebra, using τ -tilting theory.

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Theorem (Adachi-Iyama-Reiten, 2014, Marks-Stovicek, 2017)

Let Λ be a finite dimensional algebra over a field k. There are bijections between the following objects:

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Theorem (Ingalls-Thomas-type bijections)

There are bijections between the following objects:

- (1) Isomorphism classes of basic support τ -tilting modules in mod Λ .
- (1') Isomorphism classes of basic two-term silting complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.
- (2) Functorially finite torsion classes in $mod \Lambda$.
- (3) Left finite wide subcategories of $mod \Lambda$.

Theorem (Ingalls-Thomas-type bijections)

There are bijections between the following objects:

(1') Isomorphism classes of basic two-term silting complexes in K^b(proj Λ).
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given by $T \mapsto \mathcal{W}^T$ (see [Marks-Stovicek]).

Theorem (Ingalls-Thomas-type bijections)

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On the other hand, for a two-term presilting complex $U \in \mathsf{K}^{\mathsf{b}}(\mathsf{proj}\,\Lambda)$, which is a direct summand of a two-term silting complex,

 $\begin{array}{cccc} \text{two-term} & & \text{two} & (\mathcal{T}_U^+, \mathcal{F}_U^+) & & & \mathcal{W}_U \\ \text{presilting} & U \implies & \text{torsion} & \cup \mid \mid \cap & \implies & \text{wide} & & & \\ \text{complex} & & pairs & (\mathcal{T}_U^-, \mathcal{F}_U^-) & & & \mathcal{T}_U^+ \cap \mathcal{F}_U^- \end{array}$

Theorem

For a two-term presilting complex U, W_U is a semistable subcategory for a stability condition

$$\theta = \sum_X a_X \langle X, - \rangle : K_0(\mathsf{mod}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

where X runs over all indecomposable direct summands of U, $a_X \in \mathbb{R}_{>0}$ and $\langle -, - \rangle$ is the Euler form.

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Lemma

For a two-term silting complex T, there is a direct summand T' of T such that $\mathcal{W}^T = \mathcal{W}_{T'}$.

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Therefore, \mathcal{W}^T is a semistable subcategory.

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Therefore, \mathcal{W}^T is a semistable subcategory. This concludes our proof of the main theorem.

Thank you for your attention!