# Wide subcategories are semistable

based on arXiv:1705.07636

**Toshiya Yurikusa**

Nagoya university

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# Today's talk

# Wide/Semistable subcategories

- $\Lambda$ : a finite dimensional algebra over a field $k$.
- $\operatorname{mod}\Lambda$ : the category of finitely generated right $\Lambda$-modules.

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A stability condition $\theta$ on $\operatorname{mod}\Lambda$ is a linear form on $K_0(\operatorname{mod}\Lambda) \otimes_{\mathbb{Z}} \mathbb{R}$, where $K_0(\operatorname{mod}\Lambda)$ is the Grothendieck group of $\operatorname{mod}\Lambda$.

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**Definition**

A semistable subcategory of $\mathrm{mod}\,\Lambda$ is a full subcategory of $\theta$-semistable $\Lambda$-modules for some stability condition $\theta$.

# Wide/Semistable subcategories

**Lemma**

*Semistable subcategories are wide subcategories.*

In general, the converse is not true.

# Main result

We proved the following theorem.

**Theorem**

*Left finite wide subcategories of* $\operatorname{mod}\Lambda$ *are semistable subcategories.*

# Ingalls-Thomas bijections

**Theorem (Ingalls-Thomas,2009)**

*For the path algebra $kQ$ of a finite connected acyclic quiver $Q$ over a field $k$, there are bijections between the following objects:*

(1) *Isomorphism classes of basic support tilting modules in* $\mathrm{mod}(kQ)$.
(2) *Functorially finite torsion classes in* $\mathrm{mod}(kQ)$.
(3) *Functorially finite wide subcategories of* $\mathrm{mod}(kQ)$.
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It was generalized for any finite dimensional algebra, using $\tau$-tilting theory.

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**Theorem (Adachi-Iyama-Reiten,2014, Marks-Stovicek,2017)**

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# Sketch of proof

**Theorem (Ingalls-Thomas-type bijections)**

*There are bijections between the following objects:*

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(1′) *Isomorphism classes of basic two-term silting complexes in* $\mathsf{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$.

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(3) *Left finite wide subcategories of* $\operatorname{mod}\Lambda$.

# Sketch of proof

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$(3)$ *Left finite wide subcategories of* $\mathsf{mod}\,\Lambda$.

*given by* $T \mapsto \mathcal{W}^T$ *(see* [Marks-Stovicek]*).*

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*given by* $T \mapsto \mathcal{W}^T$ *(see* [Marks-Stovicek]*).*

On the other hand, for a two-term presilting complex $U \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, which is a direct summand of a two-term silting complex,

$$\text{two-term presilting complex } U \Longrightarrow \text{two torsion pairs } \begin{matrix} (\mathcal{T}_U^{+}, \mathcal{F}_U^{+}) \\ \cup\!\mid \quad \mid\!\cap \\ (\mathcal{T}_U^{-}, \mathcal{F}_U^{-}) \end{matrix} \Longrightarrow \text{wide subcategory } \begin{matrix} \mathcal{W}_U \\ :\!\shortparallel \\ \mathcal{T}_U^{+} \cap \mathcal{F}_U^{-} \end{matrix}$$

# Sketch of proof

**Theorem**

*For a two-term presilting complex $U$, $\mathcal{W}_U$ is a semistable subcategory for a stability condition*

$$\theta = \sum_X a_X \langle X, - \rangle : K_0(\mathsf{mod}\,\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

*where $X$ runs over all indecomposable direct summands of $U$, $a_X \in \mathbb{R}_{>0}$ and $\langle -, - \rangle$ is the Euler form.*

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**Lemma**

*For a two-term silting complex $T$, there is a direct summand $T'$ of $T$ such that $\mathcal{W}^T = \mathcal{W}_{T'}$.*

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Therefore, $\mathcal{W}^T$ is a semistable subcategory.

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**Lemma**

*For a two-term silting complex $T$, there is a direct summand $T'$ of $T$ such that $\mathcal{W}^T = \mathcal{W}_{T'}$.*

Therefore, $\mathcal{W}^T$ is a semistable subcategory. This concludes our proof of the main theorem.

Thank you for your attention!