Classifications of Exact Structures and Cohen-Macaulay-finite Algebras

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Outline

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1. Introduction
   - Auslander Correspondence for CM-finite IG Algebras?
2. Classifications of Exact Structures
   - Exact Categories
   - Categories of Finite Type
   - Main Results
3. Applications
   - Classification of CM-finite IG Algebras
   - Other Appications

Categories of Finite Type = Algebras

k: a field.

Proposition

There exists a bijection between:

1. *Hom-finite k-categories $\mathcal{E}$ of finite type (:⇔ categories with finitely many indecomposables).*
2. *Finite-dimensional k-algebra Γ (we call Γ an Auslander algebra of $\mathcal{E}$).*

Idea

1. Categorical properties of $\mathcal{E}$ and
2. Homological behavior of its Auslander alg Γ

should be related!

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Auslander correspondence for rep-fin. algebras

Theorem (Auslander 1971)

There exists a bijection between:

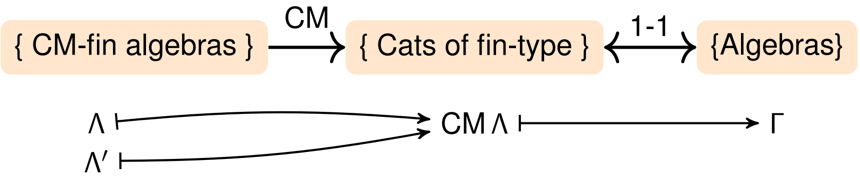
1. *Rep-fin. algebras* Λ.
2. *Abelian* k*-categories* $\mathcal{E}$ *of finite type.*
3. *Algebras* Γ *satisfying a certain homological condition* (gl.dim $\Gamma \leq 2 \leq$ dom.dim Γ).

$$\{\text{ Rep-fin algebras }\} \overset{\text{mod}}{\hookrightarrow} \{\text{ Cats of fin-type }\} \overset{\text{1-1}}{\longleftrightarrow} \{\text{Algebras}\}$$

$$\Lambda \longmapsto \mathcal{E} := \operatorname{mod}\Lambda \longmapsto \Gamma$$

$\nexists$ Auslander Correspondence for CM-fin IG Alg?

The same method doesn't work for CM-finite IG alg:



The map "CM" is not injective!
i.e. $\exists$ non-Morita-equivalent alg Λ and Λ' s.t. $\mathrm{CM}\,\Lambda \simeq \mathrm{CM}\,\Lambda'$.

However

Λ can be recovered from $\mathrm{CM}\,\Lambda$
together with the exact structure on it!

$\nexists$ Auslander Correspondence for CM-fin IG Alg?

The same method doesn't work for CM-finite IG alg:

{ CM-fin algebras } $\xrightarrow{\text{CM}}$ { Cats of fin-type } $\xleftrightarrow{\text{1-1}}$ {Algebras}

$\Lambda \mapsto \operatorname{CM}\Lambda \mapsto \Gamma$

$\Lambda' \mapsto \operatorname{CM}\Lambda$

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Λ can be recovered from $\operatorname{CM}\Lambda$
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Auslander correspondence for CM-fin algebras?

$$\left\{\begin{matrix}\text{CM-finite}\\ \text{algebras}\end{matrix}\right\} \xhookrightarrow{\text{CM}} \left\{\begin{matrix}\text{Cats of fin-type}\\ +\ \text{Exact str. on it}\end{matrix}\right\} \overset{?}{\longleftrightarrow} \left\{\begin{matrix}\text{Algebras}\\ +\ \text{some info}\end{matrix}\right\}$$

$$\Lambda \longmapsto \begin{matrix}\text{CM}\,\Lambda\text{ and}\\ \text{natural exact str.}\end{matrix} \longmapsto \Gamma$$

Our Aim

is To Construct Bijection "?" above, i.e.
To Classify exact structures on a given additive category
using its Auslander algebra.

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Exact Category

$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{E}$ is a kernel-cokernel pair
if $f = \ker g$ and $g = \operatorname{coker} f$.

Definition (Quillen 1973)

An exact category consists of a pair $(\mathcal{E}, F)$, where

- $\mathcal{E}$ is an additive category, and
- F is a class of ker-coker pairs in $\mathcal{E}$

satisfying some conditions.

Example

Λ: Iwanaga-Gorenstein alg. ($\Leftrightarrow \operatorname{id} \Lambda_\Lambda = \operatorname{id} {}_\Lambda\Lambda < \infty$),
$\mathrm{CM}\,\Lambda := \{X \in \operatorname{mod} \Lambda \mid \operatorname{Ext}_\Lambda^{>0}(X, \Lambda) = 0\}$ is naturally an exact cat.
and Λ_Λ is the progenerator (w.r.t. F).

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Auslander Algebras of Categories of Finite Type

From now on, fix a field k and

- Algebra = finite-dimensional k-algebra.
- Category = idempotent-complete Hom-finite k-category.
- $\mathcal{E}$: an idem-comp Hom-fin k-category of finite type (:$\Leftrightarrow \# \operatorname{ind} \mathcal{C}$ is finite).

Definition

An Auslander algebra Γ of $\mathcal{E}$ is defined by $\Gamma := \operatorname{End}_{\mathcal{E}}(G)$, where G is the additive generator of $\mathcal{E}$ ($\mathcal{E} = \operatorname{add} G$).

{ Cats of fin-type } $\xleftrightarrow{\text{1-1}}$ {Algebras}

$$\mathcal{E} \longmapsto \Gamma := \operatorname{End}_{\mathcal{E}}(G)$$
$$\mathcal{E} := \operatorname{proj} \Gamma \longleftarrow\!\shortmid \Gamma$$

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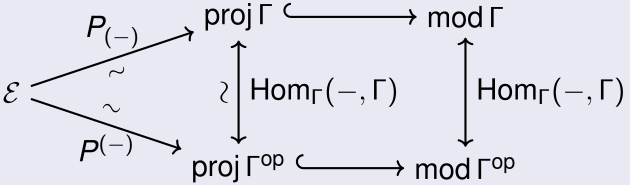
Projectivization

$\Gamma := \mathrm{End}_{\mathcal{E}}(G)$: the Auslander algebra of $\mathcal{E}$.

Proposition (Auslander's "Projectivization")



We have equivalences:

$$P_{(-)} := \mathcal{E}(G, -) : \mathcal{E} \xrightarrow{\sim} \mathrm{proj}\,\Gamma$$
$$P^{(-)} := \mathcal{E}(-, G) : \mathcal{E} \xrightarrow{\sim} \mathrm{proj}\,\Gamma^{\mathrm{op}}$$

Ker-Coker pair in $\mathcal{E}$ in terms of Γ-module

$\mathcal{E}$: cat of fin. type, Γ: its Auslander algebra.

Proposition

Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be a complex in* $\mathcal{E}$*,*
$M := \mathrm{Coker}(P_Y \to P_Z)$ *in* $\mathrm{mod}\,\Gamma$*. Then it is a ker-coker pair* $\Leftrightarrow$

1. *The following is exact in* $\mathrm{mod}\,\Gamma$ $\rightsquigarrow \mathrm{pd}\, M_\Gamma \leq 2$

$$0 \to P_X \xrightarrow{f\circ} P_Y \xrightarrow{g\circ} P_Z \to M \to 0.$$

2. *The following is exact in* $\mathrm{mod}\,\Gamma^{\mathrm{op}}$ $\rightsquigarrow \mathrm{Ext}^{0,1}_\Gamma(M, \Gamma) = 0$..

$$0 \to P^Z \xrightarrow{\circ g} P^Y \xrightarrow{\circ f} P^X \to \mathrm{Ext}^2_\Gamma(M, \Gamma) \to 0$$

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Ker-Cok pairs in $\mathcal{E} \leftrightarrow$ Objects in $\mathcal{C}_2(\Gamma)$

Definition

The subcat $\mathcal{C}_2(\Gamma) \subset \operatorname{mod}\Gamma$ consists of Γ-modules M satisfying

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$\cup$ | $\cup$

Exact str. on $\mathcal{E}$ $\longleftrightarrow$???

Main Result I

$\mathcal{E}$: cat. of fin. type, Γ: its Auslander algebra.
We have a duality $\mathrm{Ext}^2_\Gamma(-,\Gamma) : \mathcal{C}_2(\Gamma) \leftrightarrow \mathcal{C}_2(\Gamma^{\mathrm{op}})$.

Theorem (E)

There exists a bijection between the following two classes.

1. *Exact structures F on $\mathcal{E}$.*
2. *Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\Gamma)$ satisfying the following.*
    - *$\mathcal{D}$ is a Serre subcat. of $\mathrm{mod}\,\Gamma$.*
    - *$\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$ is a Serre subcat. of $\mathrm{mod}\,\Gamma^{\mathrm{op}}$.*

$\mathcal{D} \subset \mathrm{mod}\,\Gamma$ is Serre $:\Leftrightarrow$ $\mathcal{D}$ is closed under submodules, factor modules and extensions.

2-Regular Condition

Serre subcats of mod Γ $\leftrightsquigarrow$ sets of simple Γ-modules.

Definition

A simple Γ-module S is called 2-regular $:\Leftrightarrow$

1. $S \in \mathcal{C}_2(\Gamma)$, i.e, $\operatorname{pd} S_\Gamma = 2$ and $\operatorname{Ext}_\Gamma^{0,1}(S, \Gamma) = 0$.
2. $\operatorname{Ext}_\Gamma^2(S, \Gamma)$ is a simple Γ^{op}-module.

It's a "regular version" of 2-Gorenstein condition.

2-regular simple Γ-mod correspond to AR ker-coker pairs in $\mathcal{E}$:

$0 \to X \to Y \to Z \to 0$: AR ker-cok pair in $\mathcal{E}$ $\longleftrightarrow$ $0 \to P_X \to P_Y \to P_Z \to S \to 0$ 2-reg. simple Γ-mod S

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AR Quivers and Main Result II

$\mathcal{E}$: cat. of fin. type, Γ: its Auslander algebra.

Definition

The AR quiver $Q(\mathcal{E})$ of $\mathcal{E}$ is the translation quiver defined by:

- Quiver = the usual quiver of $\mathcal{E}$ (or Γ)
- $X \leftarrow\!\!-\,- Z$ if $\exists$ an AR ker-cok pair $0 \to X \to Y \to Z \to 0$ in $\mathcal{E}$.

Theorem (E)

There exists a bijection between the following classes.

1. Exact structures on $\mathcal{E}$.
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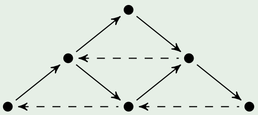
$Q(\mathcal{E})$:



$\exists$ 3 dotted arrow, hence

Red arrows are chosen.

No arrows $\leftrightarrow$ trivial exact str. of $\mathcal{E}$ (the smallest one).

All arrows $\leftrightarrow$ usual exact str. of $\mathcal{E}$ (the largest one).

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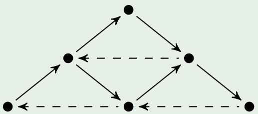

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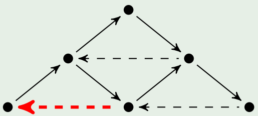

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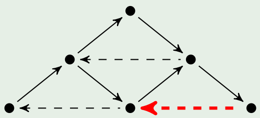

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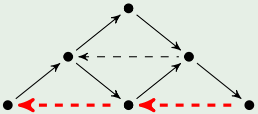

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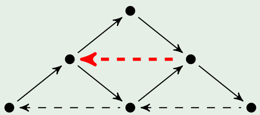

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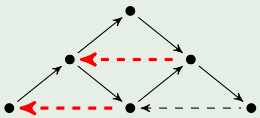

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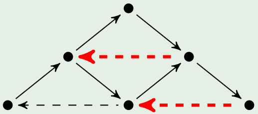

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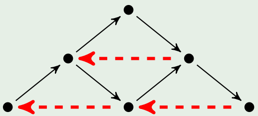

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Characterizing CM categories of IG algebras

$$\left\{\begin{array}{c}\text{CM-finite}\\ \text{IG-alg. }\Lambda\end{array}\right\} \xrightarrow[\text{CM}]{} \left\{\begin{array}{c}\text{Exact cats } \mathcal{E}\\ \text{of finite type}\end{array}\right\} \overset{\text{1-1}}{\underset{\checkmark}{\longleftrightarrow}} \left\{\begin{array}{c}\text{Alg. } \Gamma + \text{sets of}\\ \text{dotted arrows}\end{array}\right\}$$

$\mathcal{E}$: exact cat. of fin. type, Γ: its Auslander algebra.

Proposition

$\mathcal{E} \simeq \mathrm{CM}\,\Lambda$ as exact cats for some IG algebra $\Lambda \Leftrightarrow$

1. $\mathrm{gl.dim}\,\Gamma < \infty$.
2. *Projective objects in $\mathcal{E}$ = Injective objects in $\mathcal{E}$ ($\Leftrightarrow$ $\mathcal{E}$ is a Frobenius exact cat)*

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Corollary

There exists a bijection between the following.

1. CM-finite Iwanaga-Gorenstein algebras Λ.
2. Pairs $(\Gamma, \mathbb{A})$, where Γ is an algebra with $\operatorname{gl.dim} \Gamma < \infty$ and $\mathbb{A}$ is a set of dotted arrows of $Q(\operatorname{proj} \Gamma)$ which is union of stable τ-orbits (i.e. $\mathbb{A}$: disjoint union of S^1's)

$(\Gamma, \mathbb{A})$ corresponds to $\Lambda := \operatorname{End}_\Gamma(P)$, where P is the direct sum of proj. Γ-modules which are not contained in $\mathbb{A}$.

ALL CM-finite IG algebras are obtained by the following steps.

1. Take any algebra Γ with finite global dimension.
2. Draw the translation quiver $Q(\operatorname{proj} \Gamma)$.
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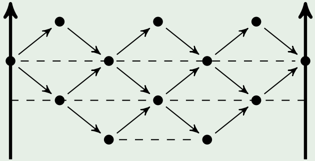
Example

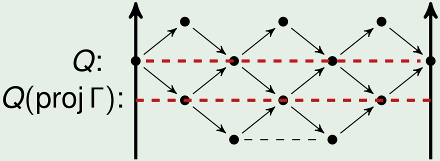
Q:



$\Gamma := kQ/$(commutativity and zero relation)
(two vertical arrows are identified).
$\Rightarrow$ the above is $Q(\mathrm{proj}\,\Gamma)$. Thus $\exists$ 2 stable τ-orbits.
$\rightsquigarrow$ We obtain $2^2 = 4$ CM-fin IG algebras Λ.

$\mathbb{A}$: Orange Dotted Arrows.
Corresponding CM-finite IG Λ is the End of Red vertices, projective object in this exact structure.

Example



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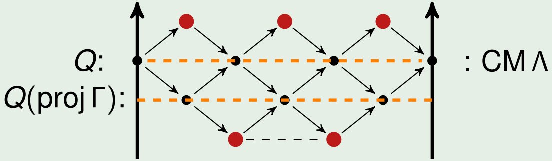

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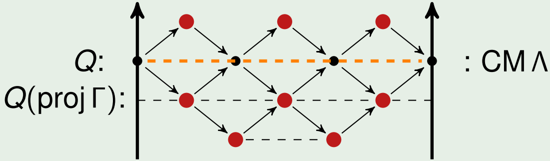

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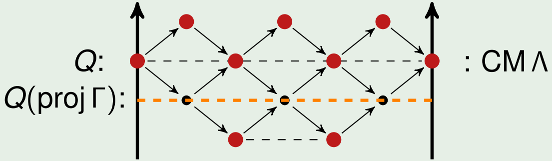

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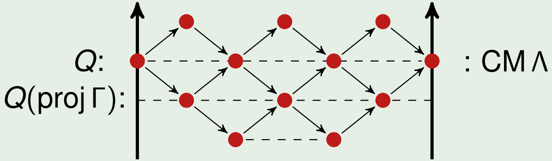

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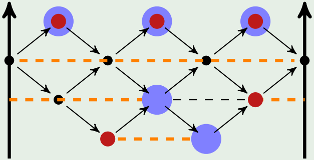
NON-Example

$Q(\operatorname{proj}\Gamma)$:



If $\mathbb{A}$: Orange, then in the corresponding exact str,

- Red: proj. objects,
- Blue: injective objects.

$\rightsquigarrow$ Proj $\neq$ Inj. (not Frobenius)
(This exact cat. is ${}^{\perp}U$ for some cotilting Λ-module U)

Other Applications

For an exact category $\mathcal{E}$ of finite type,

- $\mathcal{E}$ has enough projectives and injectives (if k is a field).
- the relation of the Grothendieck group $\mathsf{K}_0(\mathcal{E})$ is generated by AR sequences in $\mathcal{E}$.

Instead of CM-fin IG alg, a similar classification is available for cotilting Λ-modules U s.t. ${}^{\perp}U$ is of finite type.

Auslander-type correspondence for representation-finite R-orders for dim $R \geq 2$.
and so on...