The ordinary quivers of Hochschild extension algebras for self-injective Nakayama algebras

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Section 1
Hochschild extension algebras

In 2002, Fernándes and Platzeck gave the ordinary quivers of the trivial extension algebras for basic, connected and finite dimensional algebras. Moreover, they give ralations for trivial extension algebra of the algebra under the assumption that any oriented cycle in the ordinary quiver is zere. However it seems that there is little information about the ordinary quivers and relatoins for general Hochschild extension algebras.

Aim

Our aim is to describe the ordinary quivers and relations for Hochschild extension algebras for self-injective Nakayama algebras.

K : an algebraically closed field.
A : a finite dimensional K-algebra.
$D = \mathbf{Hom}_K(-, K)$: the standard duality functor.
Then $D(A) = \mathbf{Hom}_K(A, K)$ is the A-bimodule.

Definition

We define Hochschild extension over A with kernel $D(A)$ by the exact sequence

$$0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0$$

such that T is a K-algebra, ρ is an algebra homomorphism, κ is a T-bimodule monomorphism from ${}_{\rho}(D(A))_{\rho}$. Then T is called Hochschild extension algebra of A by $D(A)$.

It is well known that T is a self-injective algebra.

Definition

(F), (F') : Hochschild extensions over A with kernel $D(A)$.
(F) and (F') are equivalent if there is an algebra isomorphism $\iota : T \to T'$ such that the following diagram is commutative.

$$\begin{array}{cccccccccc} (F) & 0 & \longrightarrow & D(A) & \longrightarrow & T & \longrightarrow & A & \longrightarrow & 0 \\ & & & \| & & \downarrow \iota & & \| & & \\ (F') & 0 & \longrightarrow & D(A) & \longrightarrow & T' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

The set of all equivalence classes of Hochschild extensions is denoted by $F(A, D(A))$.

Definition

Hochschild extension

$$0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0$$

is said to be splittable if there is an algebra monomorphism $\rho' : A \to T$ with $\rho\rho' = \mathrm{id}_A$.

Fact

For an Hochschild extension algebra $T = A \oplus D(A)$, there exists a 2-cocycle $\alpha : A \times A \to D(A)$ such that the multiplication of T describes as follows:

$$(a,\, x)(b,\, y) = (ab,\, ay + xb + \alpha(a,\, b)),$$

where α is a k-bilinear map which satisfies the following condition:

$$a\alpha(b,\, c) - \alpha(ab,\, c) + \alpha(a,\, bc) - \alpha(a,\, b)c = 0$$

$(a,\, b,\, c \in A)$.

Conversely, an algebra defined as above by a 2-cocycle α is a Hochschild extension algebra. We denote the algebra by $T_\alpha(A)$.

The Hochschild (cochain) complex of A with coefficients in $D(A)$ is a sequence

$$0 \to D(A) \to \mathrm{Hom}_K(A,\, D(A)) \xrightarrow{\delta^1} \mathrm{Hom}_K(A^{\otimes 2},\, D(A)) \xrightarrow{\delta^2} \mathrm{Hom}_K(A^{\otimes 3},\, D(A)) \to \cdots \xrightarrow{\delta^{n-1}} \mathrm{Hom}_K(A^{\otimes n},\, D(A)) \xrightarrow{\delta^n} \cdots$$

$$\delta^1(\beta)(a \otimes b) = a\beta(b) - \beta(ab) + \beta(a)b,$$
$$\delta^2(\gamma)(a \otimes b \otimes c) = a\gamma(b \otimes c) - \gamma(ab \otimes c) + \gamma(a \otimes bc) - \gamma(a \otimes b)c,$$

$(\beta \in \mathrm{Hom}_K(A,\, D(A)),\ \gamma \in \mathrm{Hom}_K(A^{\otimes 2},\, D(A)),\ a,\, b,\, c \in A)$.

Definition

The group

$$H^2(A,\, D(A)) := \mathrm{Ker}\,\delta^2/\mathrm{Im}\,\delta^1 = Z^2(A,\, D(A))/B^2(A,\, D(A))$$

is called 2nd Hochschild cohomology group with coefficients in $D(A)$.

Theorem (Hochschild, 1945)

We have the following one-to-one corresponding:

$$H^2(A, D(A)) \to F(A,\ D(A))$$
$$[\alpha] \mapsto [T_\alpha(A)]$$

The equivalence class of splittable extension corresponds to the zero element of $H^2(A,\ D(A))$.

$T_0(A)$ is called the trivial extension algebra of A by $D(A)$. The multiplication of $T_0(A) = A \oplus D(A)$ is defined by

$$(a,\ x)(b,\ y) = (ab,\ ay + xb).$$

The Hochschild (chain) complex is a sequence

$$\cdots \to A^{\otimes n+2} \xrightarrow{\delta_n} A^{\otimes n+1} \to \cdots \to A^{\otimes 3} \xrightarrow{\delta_1} A^{\otimes 2} \xrightarrow{\delta_0} A \to 0$$

The differential δ_n sends $a_0 \otimes \cdots \otimes a_{n+1}$ to

$$\sum_{i=1}^{n-1} (-1)^i a_0 \otimes \cdots a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \cdots a_{n+1}$$
$$+ (-1)^{n+1} a_{n+1} a_0 \otimes a_1 \otimes \cdots a_n$$

We define the 2nd Hochschild homology by

$$HH_2(A) := \mathrm{Ker}\,\delta_1 / \mathrm{Im}\,\delta_2.$$

We have the following isomorphisms between complexes:

$$\begin{aligned} D(A^{\otimes *+1}) &\cong \mathrm{Hom}_K(A \otimes_{A^e} A^{\otimes *+2},\, K) \\ &\cong \mathrm{Hom}_{A^e}(A^{\otimes *+2},\, \mathrm{Hom}_K(A,\, K)) \\ &= \mathrm{Hom}_K(A^{\otimes *},\, D(A)). \end{aligned}$$

This induces the isomorphism

$$D(HH_2(A)) \cong H^2(A,\, D(A)).$$

Section 2

The ordinary quiver of a trivial extension

A : a basic, connected and finite dimensional K-algebra
$\Delta_A = ((\Delta_A)_0, (\Delta_A)_1, s, t)$: The ordinary quiver of A
We consider the Hochschild extension

$$0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0.$$

If we identify $D(A)$ with $\mathrm{Ker}\,\rho$, $D(A)$ is a two-sided ideal of T and $D(A)^2 = 0$. $T/D(A)$ is isomorphic to A as algebras by ρ.
The complete set of primitive orthogonal idempotent $\{e_1, \ldots, e_l\}$ of A can be lifted $\{\mathbf{e}_1, \ldots, \mathbf{e}_l\}$ of T. Therefor, we have

$$(\Delta_A)_0 = (\Delta_T)_0$$

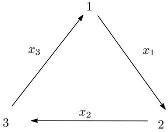
Theorem(Fernández and Platzeck, 2002)

A : a basic, connected and finite dimensional K-algebra
The ordinary quiver $\Delta_{T_0(A)}$ of trivial extension $T_0(A)$ is given by

1. $(\Delta_{T_0(A)})_0 = (\Delta_A)_0$
2. $(\Delta_{T_0(A)})_1 = (\Delta_A)_1 \cup \{\beta_{p_1}, \ldots, \beta_{p_t}\}$,

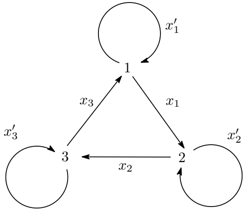
where $\{p_1, \ldots, p_t\}$ is a K-basis of $\mathrm{soc}_{A^e}(A)$ and
$\beta_{p_i} : t(p_i) \to s(p_i)$.

Example 1 $\boldsymbol{\Delta}$: the following quiver.

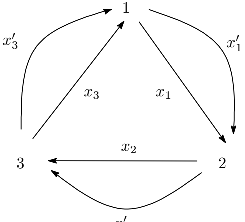


$K\Delta$: the path algebra
R_{Δ}^{n} : the two-sided ideal of $K\Delta$ generated by the paths of length n

$A := K\Delta/R_{\Delta}^{4}$
A basis of $\mathbf{soc}_{A^e}(A)$ is
$\{x_1x_2x_3,\ x_2x_3x_1,\ x_3x_1x_2\}$.
And $\Delta_{T_0(A)}$ is given by



$B := K\Delta/R_{\Delta}^{3}$
A basis of $\mathbf{soc}_{B^e}(B)$ is
$\{x_1x_2,\ x_2x_3,\ x_3x_1\}$.
And $\Delta_{T_0(B)}$ is given by



Section 3

The projective resolution by Sköldberg and one by Cibils

Theorem(Sköldberg, 1999)

$A := K\Delta/R_\Delta^n$: a truncated quiver algebra.
Δ_i : the set of paths of length i and
We have the projective resolution (P_*, d_*) of A as a left A^e-module:

$$
\begin{aligned}
P_* : \cdots &\longrightarrow A \otimes_{K\Delta_0} K\Delta_{n+1} \otimes_{K\Delta_0} A \\
&\xrightarrow{d_3} A \otimes_{K\Delta_0} K\Delta_n \otimes_{K\Delta_0} A \\
&\xrightarrow{d_2} A \otimes_{K\Delta_0} K\Delta_1 \otimes_{K\Delta_0} A \\
&\xrightarrow{d_1} A \otimes_{K\Delta_0} A \xrightarrow{d_0} A \longrightarrow 0.
\end{aligned}
$$

$$
d_2(x \otimes y_1 \cdots y_n \otimes z) = \sum_{j=0}^{n-1} x \otimes y_1 \cdots y_j \otimes y_{j+1} \otimes y_{j+2} \cdots y_n z
$$

$$
\begin{aligned}
d_3(x \otimes y_1 \cdots y_{n+1} \otimes z) = xy_1 \otimes y_2 \cdots y_{n+1} \otimes z \\
- x \otimes y_1 \cdots y_n \otimes y_{n+1} z,
\end{aligned}
$$

for $x, z \in A$ and $y_i \in \Delta_1$ $(1 \leq i \leq n+1)$.

$$\begin{aligned} A \otimes_{A^e} P_1 &= A \otimes_{A^e} (A \otimes_{K\Delta_0} K\Delta_1 \otimes_{K\Delta_0} A) \\ &\xrightarrow{\sim} A \otimes_{A^e} A^e \otimes_{K\Delta_0^e} K\Delta_1 \\ &\xrightarrow{\sim} A \otimes_{K\Delta_0^e} K\Delta_1. \end{aligned}$$

We define the degree $q(\in \mathbb{N})$ part of $A \otimes_{A^e} P_1$ by

$$\begin{aligned} (A \otimes_{A^e} P_1)_q &= (A \otimes_{K\Delta_0^e} K\Delta_1)_q \\ &:= \bigoplus_{\substack{a_i \in \Delta_1\ (1 \le i \le q) \\ \text{s.t. } t(a_{q-1}) = s(a_q), \\ t(a_q) = s(a_1)}} K(a_1 \cdots a_{q-1} \otimes a_q). \end{aligned}$$

Similarly,

$$
\begin{aligned}
(A \otimes_{A^e} P_2)_q &= (A \otimes_{K\Delta_0^e} K\Delta_n)_q \\
&:= \bigoplus_{\substack{a_i \in \Delta_1\ (1 \le i \le q) \\ \text{s.t. } t(a_{q-n}) = s(a_{q-n+1}), \\ t(a_q) = s(a_1)}} K(a_1 \cdots a_{q-n} \otimes a_{q-n+1} \cdots a_q), \\
(A \otimes_{A^e} P_3)_q &= (A \otimes_{K\Delta_0^e} K\Delta_{n+1})_q \\
&:= \bigoplus_{\substack{a_i \in \Delta_1\ (1 \le i \le q) \\ \text{s.t. } t(a_{q-n-1}) = s(a_{q-n}), \\ t(a_q) = s(a_1)}} K(a_1 \cdots a_{q-n-1} \otimes a_{q-n} \cdots a_q).
\end{aligned}
$$

Also, this grading is compatible with differential, so complex$(A \otimes_{A^e} P_*, \mathrm{id} \otimes d_*)$ is $\mathbb{N}$-graded.

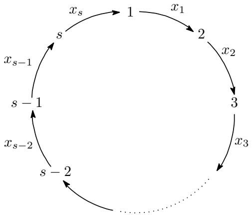
Theorem(Sköldberg, 1999)

$A := K\Delta/R_{\Delta}^{n}$: a truncated quiver algebra.
Then $HH_2(A) = \bigoplus_{q=1}^{\infty} HH_{2,q}(A)$, where

$$HH_{2,q}(A) = \begin{cases} K^{a_q} & \text{if } n+1 \leq q \leq 2n-1, \\ \bigoplus_{r|q} K^{\gcd(n,r)-1} \oplus \operatorname{Ker}(\cdot \frac{n}{\gcd(n,r)} : K \to K))^{b_r} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

Here we set $a_q := \operatorname{card}(\Delta_q^{\mathrm{c}}/C_q)$ and $b_r := \operatorname{card}(\Delta_r^{\mathrm{b}}/C_r)$.

Let Δ be the following cyclic quiver:



$A := K\Delta/R_\Delta^n$ $(n \geq 2)$. A basic and connected self-injective Nakayama algebra is isomorphic to a truncated quiver algebra of a cyclic quiver.

Theorem(Sköldberg, 1999)

$A = K\Delta/R_\Delta^n$. Then we have

$$HH_{2,q}(A) = \begin{cases} K & \text{if } s|q \text{ and } n+1 \leq q \leq 2n-1, \\ K^{s-1} \oplus \mathrm{Ker}(\cdot\frac{n}{s} : K \to K) & \text{if } s|q \text{ and } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

$$
\begin{array}{ccccc}
D(\mathrm{Tor}_2^{A^e}(A,\,A)) & & \mathrm{Ext}^2_{A^e}(A,\,D(A)) & & \\
\Vert & & \Vert & & \\
D(HH_2(A)) & \cong & H^2(A,\,D(A)) & \xleftrightarrow{\;1:1\;} & F(A,\,D(A)) \\
\wr\Vert & & & & \\
D(\bigoplus_{q=n}^{2n-1} HH_{2,\,q}(A)) & & & & \\
\wr\Vert & & & & \\
\bigoplus_{q=n}^{2n-1} D(HH_{2,\,q}(A)) & \xrightarrow{\;\exists\Theta\;} & H^2(A,\,D(A)) & &
\end{array}
$$

Theorem(Cibils, 1989)

$A = K\Delta/R^n_\Delta$: a truncated quiver algebra.
J : the Jacobson radical of A.
We have the projective resolution (Q_*, ∂_*) of A as a left A^e-module :

$$Q_* : \cdots \longrightarrow A \otimes_{K\Delta_0} J^{\otimes^3_{K\Delta_0}} \otimes_{K\Delta_0} A \xrightarrow{\partial_3} A \otimes_{K\Delta_0} J^{\otimes^2_{K\Delta_0}} \otimes_{K\Delta_0} A \xrightarrow{\partial_2} A \otimes_{K\Delta_0} J \otimes_{K\Delta_0} A \xrightarrow{\partial_1} A \otimes_{K\Delta_0} A \xrightarrow{\partial_0} A \longrightarrow 0.$$

$$\partial_2(x \otimes y_1 \otimes y_2 \otimes z) = xy_1 \otimes y_2 \otimes z - x \otimes y_1y_2 \otimes z + x \otimes y_1 \otimes y_2z,$$

$$\partial_3(x \otimes y_1 \otimes y_2 \otimes y_3 \otimes z) = xy_1 \otimes y_2 \otimes y_3 \otimes z - x \otimes y_1y_2 \otimes y_3 \otimes z + x \otimes y_1 \otimes y_2y_3 \otimes z - x \otimes y_1 \otimes y_2 \otimes y_3z,$$

for $x, z \in A$ and $y_i \in J$ $(1 \leq i \leq 3)$.

Proposition(Ames, Cagliero and Tirao, 2009)

Let x_1, x_2 be paths in Δ. We set $x_1 = a_1 a_2 \cdots a_{m_1}$, $x_2 = a_{m_1+1} a_{m_1+2} \cdots a_{m_1+m_2}$, where $a_1, a_2, \ldots, a_{m_1+m_2} \in \Delta_1$. Then there exists a map $\pi_2 : Q_2 \to P_2$ defined by the following equation:

$$\pi_2(a \otimes_{K\Delta_0} x_1 \otimes_{K\Delta_0} x_2 \otimes_{K\Delta_0} b)$$
$$= \begin{cases} a \otimes_{K\Delta_0} a_1 \cdots a_n \otimes_{K\Delta_0} a_{n+1} \cdots a_{m_1+m_2} b & \text{if } m_1 + m_2 \geq n, \\ 0 & \text{otherwise,} \end{cases}$$

for a, $b \in A$.

We have the following chain map:

$$\begin{aligned} D(A \otimes_{A^e} P_2) &\xrightarrow{D(\mathrm{id} \otimes \pi_2)} D(A \otimes_{A^e} Q_2) \\ &\to D(A \otimes_{A^e} A^{\otimes 4}) \\ &\to \mathrm{Hom}_{A^e}(A^{\otimes 4}, D(A)) \\ &\to \mathrm{Hom}_K(A^{\otimes 2}, D(A)) \end{aligned}$$

This map induces the following isomorphism:

$$\begin{aligned} \Theta : \bigoplus_{q=n}^{2n-1} D(HH_{2,q}(A)) &\cong D(\bigoplus_{q=n}^{2n-1} HH_{2,q}(A)) \\ &= D(HH_2(A)) \\ &\xrightarrow{\simeq} H^2(A, D(A)) \end{aligned}$$

Section 4
The ordinary quiver of a Hochschild extension

Lemma 1

Δ : finite quiver
$A := K\Delta/I$ for an admissible ideal I ($\exists n \geq 2$ s.t. $R_\Delta^n \subseteq I \subseteq R_\Delta^2$).
$\alpha : A \times A \to D(A)$:
2-cocycle s.t. $\forall i \in \Delta_0$, $\alpha(e_i, -) = \alpha(-, e_i) = 0$.
Then, we have the chain of subquivers of $\Delta_{T_0}(A)$:

$$\Delta \subseteq \Delta_{T_\alpha(A)} \subseteq \Delta_{T_0(A)}.$$

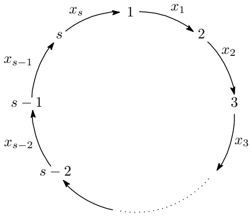
Lemma 2

Δ : finite quiver
$A := K\Delta/I$ for an admissible ideal I.
$J(A)$: the Jacobson radical of A.
$\alpha : A \times A \to D(A)$:
2-cocycle s.t. $\forall i \in \Delta_0$, $\alpha(e_i, -) = \alpha(-, e_i) = 0$,
TFAE

(1) $\alpha(J(A), J(A)) \subseteq J(A)D(A) + D(A)J(A)$.

(2) $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$.

Let $\boldsymbol{\Delta}$ be the following cyclic quiver:



Then $A := K\Delta/R_{\Delta}^{n}$ $(n \geq 2)$ is a basic and connected self-injective Nakayama algebra.

Theorem 1

$\Theta : \bigoplus_{q=1}^{2n-1} D(HH_{2,q}(A)) \to H^2(A, D(A))$.
$\alpha : A \times A \to D(A)$:
    2-cocycle s.t. $[\alpha] \in \Theta(D(HH_{2,q}(A)))$ and $[\alpha] \neq 0$.
$T_\alpha(A)$: the Hochschild extension algebra of A defined by α.
Then, we have

$$\Delta_{T_\alpha(A)} = \begin{cases} \Delta_{T_0(A)} & \text{if } n \leq q \leq 2n-2, \\ \Delta & \text{if } q = 2n-1. \end{cases}$$

Corollary 1

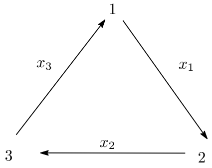
$A := K\Delta/R^n_\Delta$ $(n \geq 2)$.
$\Theta : \bigoplus_{q=1}^{2n-1} D(HH_{2,\,q}(A)) \to H^2(A,\, D(A))$.
$\alpha : A \times A \to D(A)$: 2-cocycle.
Then, $[\alpha] = \sum_{q=n}^{2n-1} [\beta_q]$, where $\beta_q : A \times A \to D(A)$ is a 2-cocycle s.t. $[\beta_q] \in \Theta(D(HH_{2,\,q}(A)))$. The following equation holds:

$$\Delta_{T_\alpha(A)} = \begin{cases} \Delta_{T_0(A)} & \text{if } [\beta_{2n-1}] = 0, \\ \Delta & \text{if } [\beta_{2n-1}] \neq 0. \end{cases}$$

Corollary 2

$A := K\Delta/R^n_\Delta$ $(n \geq 2)$.
$\alpha : A \times A \to D(A)$: 2-cocycle.
If $\Delta_{T_\alpha(A)} = \Delta$, then $T_\alpha(A)$ is isomorphic to $K\Delta/R^{2n}_\Delta$ and $T_\alpha(A)$ is symmetric.

Example 2 Let Δ be the following quiver, and we set $A := K\Delta/R_\Delta^2$.



$$\begin{aligned} HH_2(A) &= HH_{2,3}(A) \\ &= \langle (x_3 \otimes_{K\Delta_0^e} x_1x_2) + (x_1 \otimes_{K\Delta_0^e} x_2x_3) + (x_2 \otimes_{K\Delta_0^e} x_3x_1) \rangle. \end{aligned}$$

By sending the dual basis of the above basis through Θ, we have the following 2-cocycle $\alpha : A \times A \to D(A)$:

$$\alpha(a,\, b) = \begin{cases} x_i^* & \text{if } ab = x_{i+1}x_{i+2}\,(\exists i), \\ 0 & \text{otherwise}, \end{cases}$$

where $a,\, b$ are paths of length ≥ 1 and $i = 1,\, 2,\, 3$
Then we have $\Delta_{T_\alpha(A)} = \Delta$, and $T_\alpha(A) = K\Delta/R_\Delta^4$.

Section 5
Ralations for a Hochschild extension algebra

$$HH_2(A) = \bigoplus_{\substack{n \le ms \le 2n-1 \\ (m \ge 1)}} HH_{2,\,ms}(A).$$

Notice that $HH_2(A)$ has the only degree ts part $HH_{2,\,ts}(A)$ if and only if n satisfies the inequalities

$$(t-1)s < n \le ts \le 2n-1 < (t+1)s$$

for some $t \ge 1$. Then we have

$$HH_2(A) = \bigoplus_{\substack{n \le ms \le 2n-1 \\ (m \ge 1)}} HH_{2,\,ms}(A)$$

$$= \begin{cases} HH_{2,\,s}(A) & \text{if } t = 1, \text{ i.e. if } n \le s \le 2n-1, \\ HH_{2,\,2s}(A) & \text{if } t = 2, \text{ i.e. if } (n/2 \le)(2n-1)/3 < s \le n - 1/2. \end{cases}$$

Case 1: $n+1 \le s \le 2n-2$ or $(2n-1)/3 < s \le n-1$
Case 2: $s = n$

Case 1: $n+1 \le s \le 2n-2$ or $(2n-1)/3 < s \le n-1$

$$q := \begin{cases} s & \text{if } n+1 \le s \le 2n-2, \\ 2s & \text{if } (2n-1)/3 < s \le n-1. \end{cases}$$

In this case, note that
$\dim_K H^2(A, D(A)) = \dim_K HH_{2,q}(A) = 1.$

Lemma 3

If we define maps $\alpha_i : A \times A \to D(A)$ $(i = 1, \ldots, s)$ by

$$\alpha_i(a_1 \cdots a_{m_1}, a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} (\overline{x_{i+m_1+m_2} \cdots x_{i+q-1}})^* & \text{if } n \le m_1 + m_2 \le q \\ & \text{and } a_t = \overline{x_{i+t-1}} \\ & \text{for } 1 \le t \le m_1 + m_2, \\ 0 & \text{otherwise,} \end{cases}$$

then $\sum_{i=1}^{s} \alpha_i$ is a 2-cocycle, and the cohomology class $[\sum_{i=1}^{s} \alpha_i]$ is a K-basis of $H^2(A, D(A))$.

Let $\alpha = k\sum_{i=1}^{s}\alpha_i$ for $k \in K$, where α_i's are the maps in Lemma 3. Then $\Delta_{T_\alpha(A)}(= \Delta_{T_0(A)})$ is given by

1. $(\Delta_{T_\alpha(A)})_0 = (\Delta_A)_0$
2. $(\Delta_{T_\alpha(A)})_1 = (\Delta_A)_1 \cup \{x'_1, \dots, x'_s\}$,

where x'_i is an arrow from $t(p_i)$ to $s(p_i)$ corresponding to $p_i := x_{i-n+1}x_{i-n+2}\cdots x_{i-1}$ for each i $(1 \leq i \leq s)$.

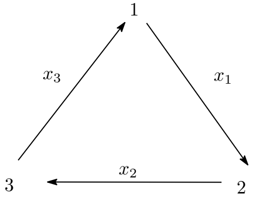
Theorem 2

Let I' be the ideal in $K\Delta_{T_\alpha(A)}$ generated by

$$x_i x'_{i+1} - x'_i x_{i-n+1}, \quad x'_i x'_{i-n+1},$$
$$x_i x_{i+1} \cdots x_{i+n-1} - k x'_i x_{i-n+1} x_{i-n+2} \cdots x_{i-n+(2n-q-1)}$$

for $i = 1, 2, \dots, s$. Then I' is admissible and $I' = I_{T_\alpha(A)}$. So $T_\alpha(A)$ is isomorphic to $K\Delta_{T_\alpha(A)}/I'$.

Example 3 Let Δ be the following quiver and we set $A := K\Delta/R_{\Delta}^{4}$.



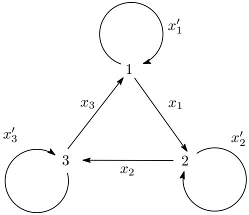
$$HH_2(A) = HH_{2,6}(A) = \langle \sum_{i=1}^{3} x_{i+4}x_{i+5} \otimes_{K\Delta_0^e} x_i x_{i+1} x_{i+2} x_{i+3} \rangle.$$

By sending the dual basis of the above basis through Θ, we have the following 2-cocycle $\alpha : A \times A \to D(A)$:

$$\alpha(a,\, b) = \begin{cases} (x_{i+4}x_{i+5})^* & \text{if } ab = x_i x_{i+1} x_{i+2} x_{i+3} \ (\exists i), \\ x_{i+5}^* & \text{if } ab = x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} \ (\exists i), \\ e_{i+6}^* & \text{if } ab = x_i x_{i+1} x_{i+2} x_{i+3} x_{i+4} x_{i+5} \ (\exists i), \\ 0 & \text{otherwise}, \end{cases}$$

where $a,\, b$ are paths of length ≥ 1 and $i = 1,\, 2,\, 3$

Then, for $k \in K$, we have $\Delta_{T_{k\alpha}(A)} = \Delta_{T_0(A)}$:



and, $T_{k\alpha}(A) = K\Delta_{T_{k\alpha}(A)}/I$, where

$$I = \langle x'_i x_i - x_i x'_{i+1},\ x_i x_{i+1} x_{i+2} x_{i+3} - k x'_i x_i,\ (x'_i)^2 \mid i = 1,\ 2,\ 3 \rangle.$$

On the other hand, $T_0(A) = K\Delta_{T_0(A)}/I_0$, where

$$I_0 = \langle x'_i x_i - x_i x'_{i+1},\ x_i x_{i+1} x_{i+2} x_{i+3},\ (x'_i)^2 \mid i = 1,\ 2,\ 3 \rangle.$$

Case 2: $s = n$
$\dim_K H^2(A,\ D(A)) = \dim_K HH_{2,\,q}(A) = s - 1$

Lemma 4

If we define maps $\alpha_i : A \times A \to D(A)$ $(i = 1, 2, \ldots, s-1)$ by

$$\alpha_i(a_1 \cdots a_{m_1},\ a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} \overline{e_i}^* & \text{if } m_1 + m_2 = s \\ & \quad \text{and } a_t = \overline{x_{i+t-1}} \\ & \quad \text{for } 1 \leq t \leq s, \\ 0 & \text{otherwise,} \end{cases}$$

then α_i's are 2-cocycles, and the set of the cohomology classes is a K-basis of $H^2(A,\ D(A))$.

Let $\alpha = \sum_{i=1}^{s-1} k_i\alpha_i$ for $k_i \in K$, where α_i's are the 2-cocycles as in Lemma 4. Then $\Delta_{T_\alpha(A)}(= \Delta_{T_0(A)})$ is given by

1. $(\Delta_{T_\alpha(A)})_0 = (\Delta_A)_0$
2. $(\Delta_{T_\alpha(A)})_1 = (\Delta_A)_1 \cup \{x'_1, \ldots, x'_s\}$,

where x'_i is an arrow from $t(p_i)$ to $s(p_i)$ corresponding to $p_i := x_{i-n+1}x_{i-n+2}\cdots x_{i-1}$ for each i $(1 \leq i \leq s)$.

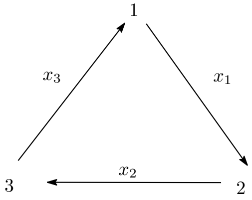
Theorem 3

Let I' be the ideal in $K\Delta_{T_\alpha(A)}$ generated by

$$x_jx'_{j+1} - x'_jx_{j+1}, \quad x'_jx'_{j+1}, \quad x_sx_1\cdots x_{s-1},$$
$$x_lx_{l+1}\cdots x_{l+s-1} - k_lx'_lx_{l+1}\cdots x_{l+s-1}$$

for $j = 1, 2, \ldots, s$ and $l = 1, 2, \ldots, s-1$. Then I' is admissible and $I' = I_{T_\alpha(A)}$. So $T_\alpha(A)$ is isomorphic to $K\Delta_{T_\alpha(A)}/I'$.

Example 4 Let Δ be the following quiver and we set $A := K\Delta/R_{\Delta}^{3}$.



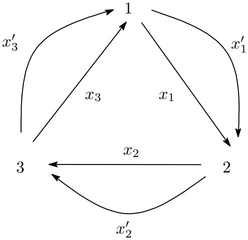
$$HH_2(A) = HH_{2,3}(A) = \langle e_1 \otimes_{K\Delta_0^e} x_1x_2x_3,\ e_2 \otimes_{K\Delta_0^e} x_2x_3x_1 \rangle.$$

By sending the dual basis of the above basis through Θ, we have the following 2-cocycle $\alpha_i : A \times A \to D(A)$:

$$\alpha_i(a,\, b) = \begin{cases} e_i^* & \text{if } ab = x_i x_{i+1} x_{i+2}, \\ 0 & \text{otherwise,} \end{cases}$$

where $a,\, b$ are paths of length ≥ 1 and $i = 1,\, 2$

For any $\mathbf{2}$-cocycle $\alpha := k_1\alpha_1 + k_2\alpha_2(k_1,\, k_2 \in K)$, we have $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$:



and, $T_\alpha(A) = K\Delta_{T_\alpha(A)}/I$, where

$$I = \langle x_i x'_{i+1} - x'_i x_{i+1},\, x'_i x'_{i+1},\\ x_1x_2x_3 - k_1x'_1x_2x_3,\, x_2x_3x_1 - k_2x'_2x_3x_1,\, x_3x_1x_2 \,|\, i = 1,\, 2,\, 3\rangle.$$

On the other hand, $T_0(A) = K\Delta_{T_0(A)}/I_0$, where

$$I_0 = \langle x_i x'_{i+1} - x'_i x_{i+1},\, x'_i x'_{i+1},\, x'_i x'_{i+1} x'_{i+2} | \, i = 1,\, 2,\, 3\rangle.$$

Outline of the proof

Lemma 1

Δ : finite quiver
$A := K\Delta/I$ for an admissible ideal I ($\exists n \geq 2$ s.t. $R_\Delta^n \subseteq I \subseteq R_\Delta^2$).
$\alpha : A \times A \to D(A)$:
  2-cocycle s.t. $\forall i \in \Delta_0$, $\alpha(e_i, -) = \alpha(-, e_i) = 0$.
Then, we have the chain of subquivers of $\Delta_{T_0}(A)$:

$$\Delta \subseteq \Delta_{T_\alpha(A)} \subseteq \Delta_{T_0(A)}.$$

Lemma 2

Δ : finite quiver
$A := K\Delta/I$ for an admissible ideal I.
$J(A)$: the Jacobson radical of A.
$\alpha : A \times A \to D(A)$:
  2-cocycle s.t. $\forall i \in \Delta_0$, $\alpha(e_i, -) = \alpha(-, e_i) = 0$,
TFAE

(1) $\alpha(J(A), J(A)) \subseteq J(A)D(A) + D(A)J(A)$.
(2) $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$.

Case 1: $n+1 \leq q \leq 2n-2$,

Case 2: $q = n$,

Case 3: $q = 2n-1$.

Case 1 $(n+1 \leq q \leq 2n-2)$

$$\dim_K HH_{2,q}(A) = 1$$

The 2-cocycle α is given by $\alpha = k\sum_{i=1}^{s} \alpha_i$ $(k \in K)$, where

$$\alpha_i(a_1 \cdots a_{m_1},\, a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} (x_{i+m_1+m_2} \cdots x_{i+q-1})^* & \text{if } n \leq m_1+m_2 \leq q \\ & \text{and } a_t = x_{i+t-1} \\ & \text{for } 1 \leq t \leq m_1+m_2, \\ 0 & \text{otherwise.} \end{cases}$$

The α satisfies the condition Lemma 2 (1). So, $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$.

Case 2 ($q = n$)
Assume $\mathbf{char}(K)|(\frac{n}{s})$ (when that is not the case, we proceed similarly.)

$$\dim_K HH_{2,q}(A) = s$$

The 2-cocycle α_i ($i = 1, \ldots, s$) is given by

$$\alpha_i(a_1 \cdots a_{m_1},\, a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} e_i^* & \text{if } m_1 + m_2 = n \\ & \text{and } a_t = x_{i+t-1} \\ & \text{for } 1 \le t \le n, \\ 0 & \text{otherwise.} \end{cases}$$

$\alpha = \sum_{i=1}^{s} k_1 \alpha_i$ satisfies the condition Lemma 2 (1). So, $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$.

Case 3 ($q = 2n - 1$)

$$\dim_K HH_{2,q}(A) = 1$$

2-cocycle α is given by $\alpha = k\sum_{i=1}^{s} \alpha_i$, where

$$\alpha_i(a_1 \cdots a_{m_1},\, a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} (x_{i+m_1+m_2} \cdots x_{i+2n-2})^* & \text{if } n \leq m_1 + m_2 \leq 2n-2 \\ & \text{and } a_t = x_{i+t-1} \\ & \text{for } 1 \leq t \leq m_1 + m_2, \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, we find the following

$$\dim_K \frac{e_i J(T_\alpha(A)) e_j}{e_i J^2(T_\alpha(A)) e_j}$$

($i,\, j \in \Delta_0$). As a result, we have $\Delta_{T_\alpha(A)} = \Delta$.

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