# The number of partial matrix rings

(joint work with Mauricio Medina-Bárcenas and Khanh Tung Nguyen)

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## 1. **Historical background for rudimentary rings** :

In 1949, T. Szele showed that there is no noncommutative division ring as the endomorphism ring of an abelian group.

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Let $R$ be a commutative ring and let $M$ be a right $R$-module.
Then $\mathrm{End}_R(M)$ is a division ring iff
$M$ is $R$-isomorphic to $Q(R/P)$ and $\mathrm{End}_R(M) \cong Q(R/P)$
where $P = r_R(M)$ and $Q(R/P)$ is the field of fractions.

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where $P = r_R(M)$ and $Q(R/P)$ is the field of fractions.

## Observation

Let $R = \mathrm{Mat}_n(\mathbb{Z})$ and $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \cdots & \mathbb{Q} \end{pmatrix}_{1\times n}$
Then $\mathrm{End}_R(M) \cong \mathbb{Q}$.

### 2. Subrings of the $\mathbf{n} \times \mathbf{n}$ full matrix ring $\mathrm{Mat}_{\mathbf{n}}(\mathbf{A})$ over $\mathbf{A}$ :

It is natural to ask, under which conditions does a subset of $\mathrm{Mat}_n(A)$ become a ring? and how can we count them?

**Example**

Let $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $R_2 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, $R_3 = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $R_4 = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$,
$R_5 = \begin{pmatrix} \mathbb{Z} & m\mathbb{Z} \\ n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $R_6 = \begin{pmatrix} \mathbb{Z} & n\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, and $R_7 = \begin{pmatrix} \mathbb{Z} & 0 \\ n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.
be rings where $m, n \in \mathbb{Z}$.

Note that consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over $R_i$.
Then $\mathrm{End}_{R_i}(M) \cong \mathbb{Q}$ where $1 \leq i \leq 7$.

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Let $R = \mathrm{Mat}_4(\mathbb{Z})$ and $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. Note that $\mathrm{End}_R(M) = \mathbb{Q}$.

Consider $\mu = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \nu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_4(\mathbb{Q})$.

Set $B = \{\mu, \nu\}$.

Then $T = \{r \in R \,|\, rB = Br\} = \left\{ \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Z} \right\}$

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**An $n \times n$ partial matrix ring** :

## Definition

An $n \times n$ partial matrix ring $\mathrm{PM}_n(A)$ over a ring $A$
is a subring of a full $n \times n$ matrix ring over $A$, with elements matrices whose entries are either elements of $A$ or 0
such that nonzero entries are independent of each other.

That is, if $\mathrm{PM}_n(A) = \sum_{(i,j)\in\mathcal{U}} e_{ij}A$ is a ring where $e_{ij}$ are matrix units and $\mathcal{U}$ is a subset of the index set $\mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$.

Note that any set of matrices is not a ring.
In general, $\sum_{(i,j)\in\mathcal{U}} e_{ij}A$ is just a subset of a full matrix ring over $A$.
Note that in a partial matrix ring $R = \mathrm{PM}_n(A)$, $\sum_{i=1}^{n} e_{ii}A \subseteq R$ (because $R$ has the unity) and not every choice of an index-pair set $\mathcal{U}$ will generate a structure closed under multiplication of matrices.

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We provide examples of partial matrix rings over the ring $\mathbb{Z}$.

## Example

(i) The set of all $\mathrm{PM}_1(\mathbb{Z})$ is $\{\mathbb{Z}\}$.

(ii) The set of all $\mathrm{PM}_2(\mathbb{Z})$ is $\left\{\begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}\right\}$.

(iii) $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$ is a subset of a $3 \times 3$ full matrix ring over $\mathbb{Z}$, but is not a partial matrix ring because $\begin{pmatrix} 1 & e & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & e & 0 \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2e & ef \\ 0 & 1 & 2f \\ 0 & 0 & 1 \end{pmatrix} \notin \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$ for $0 \neq e, f \in \mathbb{Z}$..

(iv) $\sum_{(i,j)\in\mathcal{U}} e_{ij}\mathbb{Z} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ is a partial matrix ring where $\mathcal{U} = \{(1,1),(2,2),(3,3),(4,4)\} \cup \{(1,2),(1,3),(1,4)\}$.

**Partial matrix rings v.s. Transitive digraphs** :

### Definition

Let $G = (V, E)$ be a directed graph.
$G$ is called transitive if, for the three different vertices $i, j, k \in V$, $(i, j), (j, k) \in E$ implies $(i, k) \in E$.

Note that a directed graph (shortly, digraph) disallows both multiple edges and loops.

### Theorem

*There is a bijective map from the set of all transitive directed graphs* $G = (V, E)$ *with* $V = \{1, \dots, n\}$ *to the set of all* $n \times n$ *partial matrix rings over a ring* $A$

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## Definition

A partial order is a binary relation $\mathcal{R}$ over a set $V$ satisfying the following axioms

- Reflexive: $x\mathcal{R}x$ for any $x \in V$.
- Antisymmetric: For any $x, y \in V$, if $x\mathcal{R}y$ and $y\mathcal{R}x$ then $x = y$.
- Transitive: For any $x, y, z \in V$, if $x\mathcal{R}y$ and $y\mathcal{R}z$ then $x\mathcal{R}z$.

A binary relation is called a preorder if it satisfies the reflexive axiom and the transitive axiom. Note that $x\mathcal{R}y \Leftrightarrow (x, y) \in \mathcal{R}$.

**Transitive digraphs v.s. Preorders** :

### Theorem

*There is a bijective map from the set of all transitive directed graphs* $G = (V, E)$ *with* $V = \{1, \ldots, n\}$ *to the set of all preorders on* $\{1, \ldots, n\}$*.*

### Corollary

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## Definition

An $n \times n$ Boolean matrix $B = (b_{ij})$ is a matrix such that $b_{ij} \in \{0, 1\}$.

For an $n \times n$ Boolean matrix $B = (b_{ij})$,
we define a binary relation $\leq_B$ on $\{1, 2, \ldots, n\}$ given by

$$i \leq_B j \Longleftrightarrow b_{ij} = 1.$$

## Definition

Let $B = (b_{ij})$ be an $n \times n$ Boolean matrix and $A$ any ring with unity. Consider

$$S(B, A) = \{(c_{ij}) \in \mathrm{Mat}_n(R) \mid b_{ij} = 0 \Rightarrow c_{ij} = 0\}.$$

$S(B, A)$ is a ring with unity if and only if $\leq_B$ is a preorder.
In this case $S(B, A)$ is called a structural matrix ring.

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**Partial matrix rings** v.s. **Structural matrix rings** :

It is easy to see that every structural matrix ring is a partial matrix ring. Accurately, from the definition and the previous corollary, we can directly get the following result.

### Corollary

*The $n \times n$ partial matrix rings and the $n \times n$ structural matrix rings coincide.*

## Main Theorem

*The following sets have the same cardinality:*

1. *The set of all $n \times n$ partial matrix rings over a ring.*
2. *The set of all transitive directed graphs $G = (V, E)$ with $V = \{1, \ldots, n\}$.*
3. *The set of all $n \times n$ structural matrix rings.*
4. *The set of all preorders on $\{1, \ldots, n\}$.*

### Definition

A directed graph is called strongly connected if there is a path in each direction between each pair of vertices of the graph.
In a directed graph $G$ that may not be strongly connected,
a pair of vertices $u$ and $v$ are said to be strongly connected to each other if there is a path in each direction between them.
The binary relation of being strongly connected is an equivalence relation, and its equivalence classes are called strongly connected components.

### Lemma

*Let $G = (V, E)$ be a transitive directed graph.*
*If $G$ is strongly connected, then $G$ is a complete directed graph.*

### Lemma

*Let $G = (V, E)$ be a transitive directed graph*
*and $X, Y$ be two strongly connected components.*
*If there exist $x \in X, y \in Y$ such that $(x, y) \in E$,*
*then $(x', y') \in E$ for every $x' \in X, y' \in Y$.*

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**Partial orders v.s. Transitive digraphs** :

### Proposition

*Let* $n$ *be a positive integer. Then there is a bijective map from the set of all partial orders on* $\{1, \ldots, n\}$ *to the set of all transitive directed graphs* $G = (V, E)$ *such that* $G$ *has* $n$ *strongly connected components with* $V = \{1, \ldots, n\}$.

## **The number of partial matrix rings** :

### Definition

A Stirling number of the second kind is the number of ways to partition a set of $n$ objects into $k$ non-empty subsets and is denoted by $S(n, k)$.
Note that $S(n, k) = \left\{ {k \atop j} \right\} = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$.

### Theorem

*Let $R$ be a ring and $n$ a positive integer. The number of partial matrix rings of $\mathrm{Mat}_n(R)$ is given by the following formula*

$$\sum_{k=1}^{n} S(n, k)a(k)$$

*where $S(n, k)$ is the Stirling number of the second kind and $a(k)$ is the number of partial orders on the set with $k$ elements.*

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**The number of preorders** :

Next, we can get the well-known formula, which is the relation between the number of preorders and that of partial orders as a corollary.

### Corollary

*The number of preorders on* $\{1, \ldots, n\}$ *is* $\sum_{k=1}^{n} S(n,k)a(k)$ *where* $S(n,k)$ *is the Stirling number of the second kind and* $a(k)$ *is the number of partial orders on the set with* $k$ *elements.*

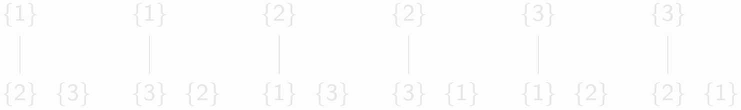
## Definition

It is called a Hasse diagram if for a partially ordered set $(S, \leq)$, one represents each element of $S$ as a vertex in the plane and draws a line segment or curve that goes upward from $x$ to $y$ whenever $y$ covers $x$ (that is, whenever $x < y$ and there is no $z$ such that $x < z < y$).

## Example

Let us draw the $3 \times 3$ partial matrix ring using a Hasse diagram: There are 29 $3 \times 3$ partial matrix rings.

$\{1\}\ \{2\}\ \{3\} \quad : \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$



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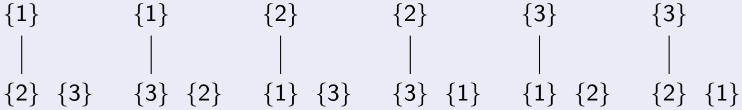
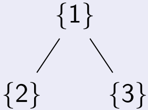


$$\begin{pmatrix} \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$$

# Example



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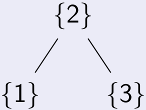
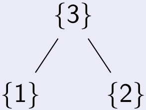
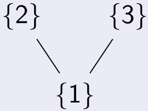


,



:

$$\begin{pmatrix} \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$$



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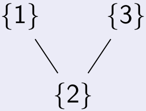

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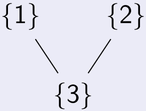

:

$$\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, \quad \begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

## Example

$$\begin{array}{cccccccccccc}
\{1\} & & \{1\} & & \{2\} & & \{2\} & & \{3\} & & \{3\} \\
| & & | & & | & & | & & | & & | \\
\{2\} & & \{3\} & & \{1\} & & \{3\} & & \{1\} & & \{2\} \\
| & & | & & | & & | & & | & & | \\
\{3\} & , & \{2\} & , & \{3\} & , & \{1\} & , & \{2\} & , & \{1\} \\
\begin{pmatrix} \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}
\end{array}$$

$$\begin{array}{cccccc}
\{1\} \quad \{2,3\} & , & \{2\} \quad \{1,3\} & , & \{3\} \quad \{1,2\} & : \\
\begin{pmatrix} \mathbb{Z} & 0 & 0 \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix} & , &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix} &
\end{array}$$

## Example

$$\begin{array}{cccccc}
\{1\} & \{2\} & \{3\} & \{2,3\} & \{1,3\} & \{1,2\} \\
| & | & | & | & | & | \\
\{2,3\}, & \{1,3\}, & \{1,2\}, & \{1\}, & \{2\}, & \{3\} \\
\begin{pmatrix} \mathbb{Z} & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}, &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}, &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{pmatrix}, &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \end{pmatrix}, &
\begin{pmatrix} \mathbb{Z} & 0 & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & 0 & \mathbb{Z} \end{pmatrix}, &
\begin{pmatrix} \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}
\end{array}$$

$$\{1,2,3\} \quad : \quad \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}.$$

A. The number of partial orders $a(n)$ on the finite set $\{1, \ldots, n\}$:

- $a(0) = 1$
- $a(1) = 1$
- $a(2) = 3$
- $a(3) = 19$
- $a(4) = 219$
- $a(5) = 4231$
- $a(6) = 130023$
- $a(7) = 6129859$
- $a(8) = 431723379$
- $a(9) = 44511042511$
- $a(10) = 6611065248783$
- $a(11) = 1396281677105899$
- $a(12) = 414864951055853499$
- $a(13) = 171850728381587059351$
- $a(14) = 98484324257128207032183$
- $a(15) = 77567171020440688353049939$
- $a(16) = 83480529785490157813844256579$
- $a(17) = 122152541250295322862941281269151$
- $a(18) = 241939392597201176602897820148085023$

B. The number of partial matrix rings $p(n)$ of $M_n(R)$:

- $p(1) = 1$
- $p(2) = 4$
- $p(3) = 29$
- $p(4) = 355$
- $p(5) = 6942$
- $p(6) = 209527$
- $p(7) = 9535241$
- $p(8) = 642779354$
- $p(9) = 63260289423$
- $p(10) = 8977053873043$
- $p(11) = 1816846038736192$
- $p(12) = 519355571065774021$
- $p(13) = 207881393656668953041$
- $p(14) = 115617051977054267807460$
- $p(15) = 88736269118586244492485121$
- $p(16) = 93411113411710039565210494095$
- $p(17) = 134137950093337880672321868725846$
- $p(18) = 261492535743634374805066126901117203$

We assume $n \geq 2$. For a matrix $\mu$, $\mu^t$ will stand for the transpose of $\mu$.

### Lemma

*Let S be a division ring. Consider an integral domain A with* $Q(A) = S$. *Let* $R = \mathrm{PM}_n(A)$. *Assume that M is a faithful right R-module with* $S = End_R(M)$ *and* $dim_S(M) = n$ *for some* $n \in \mathbb{N}$. *For any* $\mu \in \mathrm{PM}_n(S)$, *set* $T = \{r \in R \,|\, r\mu = \mu r\}$.
*Then* $End_T(M) \supseteq S[\mu^t]$.

### Question

*Let S be a division ring. Consider an integral domain A with* $Q(A) = S$. *Let* $R = \mathrm{PM}_n(A)$. *Assume that M is a faithful right R-module with* $S = End_R(M)$ *and* $dim_S(M) = n$ *for some* $n \in \mathbb{N}$. *For any* $\mu \in \mathrm{PM}_n(S)$, *set* $T = \{r \in R \,|\, r\mu = \mu r\}$.
*When does* $End_T(M) = S[\mu^t]$ *hold true?*

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**Example**

Let $R = \mathrm{M}_2(\mathbb{Z})$ and $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. Then $\mathrm{End}_R(M) \cong \mathbb{Q}$.
Consider $\mu = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Q})$. Then
$T = \{r \in R \mid r \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right) r\} = \left\{ I_2 a + \left(\begin{smallmatrix} 0 & 2 \\ 3 & 3 \end{smallmatrix}\right) b \mid a, b \in \mathbb{Z} \right\}$ and
$\mathrm{End}_T(M) = \mathbb{Q}[\mu^t]$.

**Example**

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### Example

Let $R = \mathrm{M}_3(\mathbb{Z})$ and $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. Then $\mathrm{End}_R(M) \cong \mathbb{Q}$.
Consider $\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \in \mathrm{M}_3(\mathbb{Q})$. Then
$T = \{ r \in R \mid r \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} r \} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$ and
$\mathrm{End}_T(M) = \{ I_3\alpha + e_{11}\beta + e_{22}\gamma \mid \alpha, \beta, \gamma \in \mathbb{Q} \} = \mathbb{Q}[\mu^t]$.

### Example

Let $R = \mathrm{M}_3(\mathbb{Z})$ and $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$. Then $\mathrm{End}_R(M) \cong \mathbb{Q}$.
Consider $\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{M}_3(\mathbb{Q})$. Then
$T = \{ r \in R \mid r \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} r \} = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} \mid a, b, c, d, e \in \mathbb{Z} \right\}$
and $\mathrm{End}_T(M) = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \right\} \mid \alpha, \beta \in \mathbb{Q} \} = \mathbb{Q}[\mu^t]$.

## Example

Let $R = \mathrm{M}_3(\mathbb{Z})$ and $M = (\,\mathbb{Q}\ \mathbb{Q}\ \mathbb{Q}\,)$. Then $\mathrm{End}_R(M) \cong \mathbb{Q}$.
Consider $\mu = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right) \in \mathrm{M}_3(\mathbb{Q})$. Then
$T = \{r \in R \,|\, r\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right) r\} = \left\{ \left(\begin{smallmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{smallmatrix}\right) \,|\, a, b, c \in \mathbb{Z} \right\}$ and
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and $\mathrm{End}_T(M) = \left\{ \left(\begin{smallmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{smallmatrix}\right) \right\} \,|\, \alpha, \beta \in \mathbb{Q}\} = \mathbb{Q}[\mu^t]$.

Thank you

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- L. van Wyk, Maximal left ideals in structural matrix rings, Comm. Algebra, **1988** *16(2)*, 399–419