Complex Rings, Quaternion Rings and Octonion Rings

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In 1840s, Hamilton discovered quaternions and Kelly, Graves independently discovered octonions. These numbers are defined over real numbers and contain complex numbers. Through Frobenius, Wedderburn, these numbers have been studied by many mathematicians. We may say the roots of our ring and representation theory began with these numbers.

In oder to define these numbers for any ring R, we consider free right R-modules:

$$\boldsymbol{C}(R) = e_0R \oplus e_1R,$$
$$\boldsymbol{H}(R) = e_0R \oplus e_1R \oplus e_2R \oplus e_3R,$$
$$\boldsymbol{O}(R) = e_0R \oplus e_1R \oplus \cdots \oplus e_7R.$$

We define $re_i = e_ir$ for $\forall r \in R, \forall e_i$ and multiplication for $\{e_i\}_i$ by the following Cayley-Graves multiplication table:

$\times$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\mathbf{e}_1$	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
$\mathbf{e}_2$	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
$\mathbf{e}_3$	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
$\mathbf{e}_4$	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
$\mathbf{e}_5$	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
$\mathbf{e}_6$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
$\mathbf{e}_7$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

Then $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ are rings, and $\boldsymbol{O}(R)$ is a non-associative ring. We call $\boldsymbol{C}(R)$ a complex ring, $\boldsymbol{H}(R)$ a quaternion ring and $\boldsymbol{O}(R)$ an octonion ring. For $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$, we put $1 = e_0, i = e_1, j = e_2, k = e_3$. Then multiplication for $\{i, j, k\}$ are usual forms:

$$i^2 = j^2 = k^2 = -1,$$
$$ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

In order to study $\boldsymbol{H}(\boldsymbol{H}(R))$, we use $\{\mathbb{i}, \mathbb{j}, \mathbb{k}\}$ instead of $\{i, j, k\}$. Namely,

$$\boldsymbol{H}(R) = R + iR + jR + kR,$$
$$\boldsymbol{H}(\boldsymbol{H}(R)) = \boldsymbol{H}(R) + \mathbb{i}\boldsymbol{H}(R) + \mathbb{j}\boldsymbol{H}(R) + \mathbb{k}\boldsymbol{H}(R).$$

Similarly, for $\boldsymbol{C}(\boldsymbol{H}(R)), \boldsymbol{C}(\boldsymbol{C}(R)), \boldsymbol{H}(\boldsymbol{C}(R))$, we use $\mathbb{i}, \mathbb{j}, \mathbb{k}$.

In view of progress of quaternion rings, generalized quaternion rings were introduced for commutative fields and these rings have been extensively studied. For later use, we introduce generalized quaternion rings over any ring R.

Let R be a ring and let a, b be non-zero elements of the center of R. For a free right R-module $M = R \oplus iR \oplus jR \oplus kR$, we define

$$ri = ir,\ rj = jr,\ rk = kr \quad \forall r \in R$$

and multiplication for $\{i, j, k\}$ as follows:

$$i^2 = a,\ j^2 = b,\ ij = -ji = k.$$

Then we can see the following:

$$k^2 = -ab,\ ik = -ki = ja,\ jk = -kj = -ib.$$

By this multiplication, M is a ring. We denote M by

$$\left(\frac{a, b}{R}\right)$$

For a commutative field F, $\left(\frac{a,b}{F}\right)$ is well studied as number theory. In our talk, for a division ring D, we show certain results on $\left(\frac{a,b}{D}\right)$, from which we know the difference between $\left(\frac{a,b}{F}\right)$ and $\left(\frac{a,b}{D}\right)$. For $\left(\frac{a,b}{F}\right)$, three books below are nice references.

[1] T. Y. Lam, The Algebraic Theory of Quadratic Forms, Reading: Addison Wesley-Benjamin (1973).

[2] R. S. Pierce, Associative Algebras, Springer-Verlag, New York, Heidelberg, Berlin (1982).

[3] S. Saito, Seisuuron, Kyoritsu Shuppan (1997). (斉藤秀司, 整数論, 共立出版.)

Symbols

$\mathbb{R}$	real numbers
$M_n(R)$	$n \times n$ matrix ring over a ring R
$J(R)$	Jacobson radical of R
$S(R_R)$	Socle of R_R
$\lvert X \rvert$	cardinality of a set X
$Pi(R)$	complete set of orthogonal primitive idempotents of an artinian ring R

Now, recently, Lee-O. showed the following results (Frontiers of Mathematics in China):

(A) If R is a Frobenius algebra, then $\boldsymbol{C}(R), \boldsymbol{H}(R)$ and $\boldsymbol{O}(R)$ are Frobenius algebras.

(B) If R is a quasi-Frobenius ring, then $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ are quasi-Frobenius rings.

It follows from (B) that, for a division ring D, $\boldsymbol{C}(D)$ and $\boldsymbol{H}(D)$ are quasi-Frobenius rings. One of main results in our talk is the following:

Theorem 1

When $2 \neq 0$, $\boldsymbol{C}(D)$ and $\boldsymbol{H}(D)$ are simple rings such that

1. $\boldsymbol{C}(D)$ *is a division ring or* $|Pi(\boldsymbol{C}(D))| = 2$.
2. $\boldsymbol{H}(D)$ *is a division ring or* $|Pi(\boldsymbol{C}(D))| = 2$ *or* 4.

Let R be a quasi-Frobenius ring. In order to study the structure of $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$, we first observe idempotents and nilpotents in these rings. For $\alpha = a + ib + jc + kd \in \boldsymbol{H}(R)$, we write

$$\alpha^2 = A + iB + jC + kD$$

where $a, b, c, d, A, B, C, D \in R$. Then, by calculation, we see

$$A = a^2 - b^2 - c^2 - d^2$$
$$B = ba + ab + cd - dc$$
$$C = ca + ac + db - bd$$
$$D = da + ad + bc - cb$$

$$\therefore \quad \alpha^2 = 0$$

$$\Longleftrightarrow$$

$$(\#) \begin{cases} a^2 - b^2 - c^2 - d^2 & = & 0 \\ ba + ab + cd - dc & = & 0 \\ ca + ac + db - bd & = & 0 \\ da + ad + bc - cb & = & 0 \end{cases}$$

Further,

$$\alpha^2 = \alpha$$

$$\iff$$

$$(*) \begin{cases} a^2 - b^2 - c^2 - d^2 &= a \\ ba + ab + cd - dc &= b \\ ca + ac + db - bd &= c \\ da + ad + bc - cb &= d \end{cases}$$

By $(*)$, we obtain

Fact 1

F : field, $2 \neq 0$. Then

$$\alpha^2 = \alpha \iff a = \frac{1}{2}, \quad \frac{1}{4} + b^2 + c^2 + d^2 = 0$$

By $(*)$, we can show the following:

Theorem 2

1. $J(\boldsymbol{H}(D)) = 0, \quad \boldsymbol{H}(D)$: *indecomposable simple ring*
2. $|Pi(\boldsymbol{H}(D))| = 1$ *or* 2 *or* 4
3. $|Pi(\boldsymbol{H}(D))| = 1 \iff \boldsymbol{H}(D)$: *division ring*
4. $|Pi(\boldsymbol{H}(D))| = 2 \implies \forall$ *primitive idempotent* $e \in \boldsymbol{H}(D)$,

$$\boldsymbol{H}(D) \cong \begin{pmatrix} e\boldsymbol{H}(D)e & e\boldsymbol{H}(D)e \\ e\boldsymbol{H}(D)e & e\boldsymbol{H}(D)e \end{pmatrix}$$

5. $D = F$: *commutative field*

$$\Longrightarrow \quad \boldsymbol{H}(F): \text{ division ring} \quad \text{or} \quad \boldsymbol{H}(F) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$$

6. *For a commutative field* F, $|Pi(\boldsymbol{H}(F))| = 4$ *does not occur.*

We shall give a sketch of the proof of (1) in this theorem. We may show the following:

Lemma 3

Let $\alpha \in \boldsymbol{H}(D)$. Then

$$\alpha^2 = (\alpha i)^2 = (\alpha j)^2 = (\alpha k)^2 = 0 \;\Rightarrow\; \alpha = 0.$$

(Proof) Let $\alpha = a + ib + jc + kd \in \boldsymbol{H}(D)$. By $\alpha^2 = 0$ and $(\#)$,

$$a^2 - b^2 - c^2 - d^2 = 0 \tag{1}$$

Since $\alpha i = -b + ia + jd - kc$ and $(\alpha i)^2 = 0$,

$$b^2 - a^2 - d^2 - c^2 = 0 \tag{2}$$

Similarly, by $(\alpha j)^2 = 0$ and $(\alpha k)^2 = 0$,

$$c^2 - d^2 - a^2 - b^2 = 0 \tag{3}$$

$$d^2 - c^2 - b^2 - a^2 = 0 \tag{4}$$

By (1) + (2),

$$-2c^2 - 2d^2 = 0.$$
$$\therefore c^2 + d^2 = 0$$
$$\therefore b^2 = a^2$$

By (1) + (3), $c^2 = a^2$.
By (1) + (4), $d^2 = a^2$.

$$\therefore a^2 = b^2 = c^2 = d^2$$

Since $a^2 - b^2 - c^2 - d^2 = 0$, $2a^2 = 0$

$$\therefore a = 0$$
$$\therefore a = b = c = d = 0$$
$$\therefore \alpha = 0$$

Theorem 4

Let D be a division ring with $\underline{2 = 0}$.

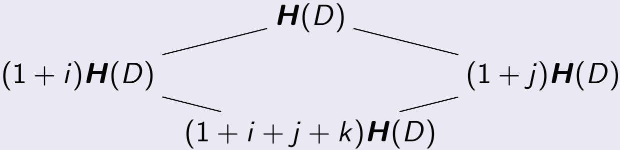
1. $\boldsymbol{C}(D)$: *local quasi-Frobenius ring s.t.*

$$J(\boldsymbol{C}(D)) = S(\boldsymbol{C}(D)) = e\boldsymbol{C}(D)$$

   where $e = 1 + i$.

2. $\boldsymbol{H}(D)$: *local quasi-Frobenius ring s.t.*

$$J(\boldsymbol{H}(D)) = (1+i)\boldsymbol{H}(D) + (1+j)\boldsymbol{H}(D)$$
$$S(\boldsymbol{H}(D)) = (1+i+j+k)\boldsymbol{H}(D)$$



Example 5

$$D := \boldsymbol{H}(\mathbb{R}) = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R} : \text{ division ring}$$

Consider

$$\boldsymbol{H}(D) = \boldsymbol{H}(\boldsymbol{H}(\mathbb{R})) = \boldsymbol{H}(\mathbb{R}) \oplus \mathbb{i}\boldsymbol{H}(\mathbb{R}) \oplus \mathbb{j}\boldsymbol{H}(\mathbb{R}) \oplus \mathbb{k}\boldsymbol{H}(\mathbb{R}).$$

Then,

$$|Pi(\boldsymbol{H}(D))| = 4.$$

In fact, put

$$g_1 = \frac{1}{4}(1 + \mathbb{i}i + \mathbb{j}j + \mathbb{k}k), \quad g_2 = \frac{1}{4}(1 + \mathbb{i}i - \mathbb{j}j - \mathbb{k}k),$$
$$g_3 = \frac{1}{4}(1 - \mathbb{i}i - \mathbb{j}j + \mathbb{k}k), \quad g_4 = \frac{1}{4}(1 - \mathbb{i}i + \mathbb{j}j - \mathbb{k}k).$$

Then, $\{g_1, g_2, g_3, g_4\}$ are orthogonal primitive idempotents.

$$\therefore |Pi(\boldsymbol{H}(D))| = 4.$$

Further, we obtain for a dividion ring D with $2 \neq 0$, the following result:

Theorem 6

If $\boldsymbol{H}(D)$ is a division ring, then

$$\boldsymbol{H}(\boldsymbol{H}(D)) \cong \begin{pmatrix} D & D & D & D \\ D & D & D & D \\ D & D & D & D \\ D & D & D & D \end{pmatrix}.$$

In particular,

$$\boldsymbol{H}(\boldsymbol{H}(\mathbb{R})) \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{pmatrix}.$$

This example and the theorem show that the worlds of $\boldsymbol{H}(D)$ and $\boldsymbol{H}(F)$ are different. About $|Pi(\boldsymbol{H}(D))| = 4$, following unexpected result holds:

Theorem 7

Let D be a division ring with $2 \neq 0$. Following conditions are equivalent:

1. $|Pi(\boldsymbol{H}(D))| = 4$.
2. $\exists p, q, r \in D$ *s.t.* $p^2 = -1, q^2 = -1, pq = r = -qp$.

Here we shall state about a generalized quaternion ring $\left(\frac{a,b}{D}\right)$ where D is a division ring with $2 \neq 0$.

The following results hold:

Theorem 8

Following conditions are equivalent:

1. $|Pi(\left(\frac{a,b}{D}\right))| = 4.$
2. $\exists p, q, r \in D$ *s.t.* $p^2 = a, q^2 = b, pq = r = -qp.$

In these case, following $\{g_i\}_i$ *are orthogonal primitive idempotents:*

$$g_1 = \frac{1}{4}(1 + ipa^{-1} + jqb^{-1} + kr(ab)^{-1}),$$
$$g_2 = \frac{1}{4}(1 + ipa^{-1} - jqb^{-1} - kr(ab)^{-1}),$$
$$g_3 = \frac{1}{4}(1 - ipa^{-1} + jqb^{-1} - kr(ab)^{-1}),$$
$$g_4 = \frac{1}{4}(1 - ipa^{-1} - jqb^{-1} + kr(ab)^{-1}).$$

We skip to state the structures of $\boldsymbol{H}(\boldsymbol{H}(D))$, $\boldsymbol{H}(\boldsymbol{C}(D))$, $\boldsymbol{C}(\boldsymbol{C}(D))$ etc. Finally we comment a classical theorem on $\boldsymbol{H}(F)$, where F is a field with $2 \neq 0$.

Following are equivalent:

(1) $\boldsymbol{H}(F) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.

(2) $1 + X^2 + Y^2 + Z^2 = 0$ has a solution.

(3) $1 + X^2 + Y^2 = 0$ has a solution.

But the following condition is **not** equivalent to these conditions.

(4) $1 + X^2 = 0$ has a solution.

Following conditions (2'), (3'), (4') correspond to (2), (3), (4), respectively.

(2') $\exists$idempotent $e = \frac{1}{2} + iX + jY + kZ \in \boldsymbol{H}(F)$.

(3') $\exists$idempotent $e = \frac{1}{2} + iX + jY \in \boldsymbol{H}(F)$.

(4') $\exists$idempotent $e = \frac{1}{2} + iX \in \boldsymbol{H}(F)$.

Instead of $\left(\frac{a,b}{D}\right)$, we may use $H(D; a, b)$.
Several questions arise: Structure of $H(H(D; -1, 1); -1, -1)$, $H(H(D; 1, -1); 1, 1)$, $H(\boldsymbol{C}(D); 1, 1), \ldots$?

Example 1

$$H(H(\mathbb{R}; -1, -1); -1, -1) \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{pmatrix}.$$