HOW TO CAPTURE t-STRUCTURES BY SILTING THEORY

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Abstract. In this note, we study a relationship between silting objects and t-structures. We introduce the notion of ST-pairs of thick subcategories of a given triangulated category, a prototypical example of which is the pair of the bounded homotopy category and the bounded derived category of a finite-dimensional algebra. For an ST-pair (C, D), we construct an injective map from silting objects in C to bounded t-structures on D, and show that the map is bijective if and only if C is silting-discrete. Moreover, using cluster tilting theory, we give a new class of silting-discrete triangulated categories.

This is based on a joint work with Dong Yang [3].

Throughout this note, K is a field and T is a K-linear Hom-finite Krull–Schmidt triangulated category with shift functor [1].

Our aim of this note is to give a construction of bounded t-structures by silting objects. First we recall the notion of t-structures, which was introduced by Beilinson–Bernstein–Deligne [8].

Definition 1. A *t-structure* on T is a pair $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ of strictly full subcategories of T such that

(1) $\mathsf{T}^{\leq 1} \supseteq \mathsf{T}^{\leq 0}$ and $\mathsf{T}^{\geq 0} \supseteq \mathsf{T}^{\geq 1}$,
(2) $\mathrm{Hom}_{\mathsf{T}}(X, Y) = 0$ for all $X \in \mathsf{T}^{\leq 0}$ and $Y \in \mathsf{T}^{\geq 1}$,
(3) for each $Z \in \mathsf{T}$, there is a triangle $X \to Z \to Y \to X[1]$ in T with $X \in \mathsf{T}^{\leq 0}$ and $Y \in \mathsf{T}^{\geq 1}$.

Here, for any integer n, let $\mathsf{T}^{\leq n} = \mathsf{T}^{\leq 0}[-n]$ and $\mathsf{T}^{\geq n} = \mathsf{T}^{\geq 0}[-n]$.

Let $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ be a t-structure on T. Then the *heart* $\mathsf{T}^0 := \mathsf{T}^{\leq 0} \cap \mathsf{T}^{\geq 0}$ is an abelian category. We call $(\mathsf{T}^{\leq 0}, \mathsf{T}^{\geq 0})$ a *bounded* t-structure if

$$\mathsf{T} = \bigcup_{n \in \mathbb{Z}} \mathsf{T}^{\leq n} = \bigcup_{n \in \mathbb{Z}} \mathsf{T}^{\geq n},$$

or equivalently, $\mathsf{T} = \mathsf{thick}\,\mathsf{T}^0$. We denote by $t\text{-str}_{bd}\mathsf{T}$ the set of bounded t-structures on T.

We give an example of bounded t-structures.

Example 2. Let Λ be a finite-dimensional algebra and $\mathsf{D} := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$ the bounded derived category. We define two full subcategories as follows:

$$\mathsf{D}^{\leq 0} := \{X \in \mathsf{D} \mid H^n X = 0 \text{ for all integers } n > 0\},$$
$$\mathsf{D}^{\geq 0} := \{X \in \mathsf{D} \mid H^n X = 0 \text{ for all integers } n < 0\}.$$

Then it is well-known that $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a bounded t-structure on D.

The detailed version of this note will be submitted for publication elsewhere.

Next we recall the definition of silting objects, which was introduced by Keller–Vossieck [17]. For details, we refer to [5].

Definition 3. An object M of T is said to be *silting* if $\mathrm{Hom}_{\mathsf{T}}(M, M[n]) = 0$ for all integers $n > 0$ and $\mathsf{T} = \mathsf{thick}M$. We denote by $\mathrm{silt}\,\mathsf{T}$ the set of isomorphism classes of basic silting objects of T.

We give a typical example of a silting object.

Example 4. Let Λ be a finite-dimensional algebra. Then Λ is a silting object of the bounded homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

We introduce the notion of ST-pairs, which plays a central role in this note. For an object M of T, we define full subcategories of T as follows:

$$\begin{aligned}
\mathsf{T}_M^{\leq 0} &:= \{X \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(M, X[n]) = 0 \text{ for all integers } n > 0\},\\
\mathsf{T}_M^{\geq 0} &:= \{X \in \mathsf{T} \mid \mathrm{Hom}_{\mathsf{T}}(M, X[n]) = 0 \text{ for all integers } n < 0\},\\
\mathsf{T}_M^{0} &:= \mathsf{T}_M^{\leq 0} \cap \mathsf{T}_M^{\geq 0}.
\end{aligned}$$

Definition 5. Let C and D be thick subcategories of T. The pair (C, D) is called an *ST-pair* inside T if there exists a silting object M of C such that

(ST1) $(\mathsf{T}_M^{\leq 0}, \mathsf{T}_M^{\geq 0})$ is a t-structure on T,
(ST2) $\mathsf{T}_M^{\geq 0} \subseteq \mathsf{D}$,
(ST3) $\mathsf{D} = \mathsf{thick}\,\mathsf{T}_M^{0}$.

When there is a need to emphasise the silting object M, we call the triple $(\mathsf{C}, \mathsf{D}, M)$ an *ST-triple*.

The following two examples are our motivating examples.

Example 6. Let Λ be a finite-dimensional algebra and $\mathsf{T} := \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$. Then we have

$$\begin{aligned}
\mathsf{T}_\Lambda^{\leq 0} &= \{X \in \mathsf{T} \mid H^n X = 0 \text{ for all integers } n > 0\},\\
\mathsf{T}_\Lambda^{\geq 0} &= \{X \in \mathsf{T} \mid H^n X = 0 \text{ for all integers } n < 0\}.
\end{aligned}$$

We obtain that Λ is a silting object of $\mathsf{C} := \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ and $(\mathsf{T}_\Lambda^{\leq 0}, \mathsf{T}_\Lambda^{\geq 0})$ is a (bounded) t-structure on T. Thus $(\mathsf{C} = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda), \mathsf{D} := \mathsf{T} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda))$ is an ST-pair inside T.

Example 7. Let Γ be a dg algebra satisfying the following conditions:

(1) $H^n(\Gamma) = 0$ for each integer $n > 0$,
(2) $H^0(\Gamma)$ is finite-dimensional,
(3) $\mathsf{D}_{\mathrm{fd}}(\Gamma) \subseteq \mathsf{per}(\Gamma)$, where $\mathsf{per}(\Gamma)$ is the perfect derived category of Γ and $\mathsf{D}_{\mathrm{fd}}(\Gamma)$ is the full subcategory of the derived category $\mathsf{D}(\Gamma)$ consisting of dg Γ-modules whose total cohomology is finite-dimensional.

Let $\mathsf{T} := \mathsf{per}(\Gamma)$. Then T is Hom-finite Krull–Schmidt by [14, Proposition 2.5], $(\mathsf{T}_\Gamma^{\leq 0}, \mathsf{T}_\Gamma^{\geq 0})$ is a t-structure on T and $\mathsf{T}_\Gamma^{\geq 0} \subseteq \mathsf{thick}\,\mathsf{T}_\Gamma^{0}$ (see [7, Proposition 2.7] and [14, Propositions 2.5 and 2.1(c)]). Since Γ is a non-positive dg algebra, Γ is a silting object of T. Moreover, we have $\mathsf{D}_{\mathrm{fd}}(\Gamma) = \mathsf{thick}\,\mathsf{T}_\Gamma^{0}$. Thus $(\mathsf{C} := \mathsf{T} = \mathsf{per}(\Gamma), \mathsf{D} := \mathsf{D}_{\mathrm{fd}}(\Gamma))$ is an ST-pair inside T.

Fix an ST-pair (C,D). For a silting object M of C, we define full subcategories of D as follows:

$$\begin{aligned} \mathsf{D}_M^{\leq 0} &:= \mathsf{T}_M^{\leq 0} \cap \mathsf{D} = \{X \in \mathsf{D} \mid \mathrm{Hom}_{\mathsf{T}}(M, X[n]) = 0 \text{ for all integers } n > 0\}, \\ \mathsf{D}_M^{\geq 0} &:= \mathsf{T}_M^{\geq 0} \cap \mathsf{D} = \mathsf{T}_M^{\geq 0}, \\ \mathsf{D}_M^{0} &:= \mathsf{D}_M^{\leq 0} \cap \mathsf{D}_M^{\geq 0}. \end{aligned}$$

The following proposition implies that the conditions (ST1–3) are satisfied for all silting objects of C, which allows us to define a well-defined map from silting objects in C to bounded t-structures on D.

Proposition 8. *Let $(\mathsf{C},\mathsf{D},M)$ be an ST-triple and let N be an arbitrary silting object of C. Then the following statements hold.*

(1) $(\mathsf{C},\mathsf{D},N)$ *is an ST-triple.*
(2) $\mathsf{T}_N^0 \simeq \mathsf{mod}\,\mathrm{End}_{\mathsf{T}}(N)$.
(3) $(\mathsf{D}_N^{\leq 0},\mathsf{D}_N^{\geq 0})$ *is a bounded t-structure on D and $\mathsf{D}_N^0 = \mathsf{T}_N^0$.*

The following theorem is one of our main results in this note.

Theorem 9. *Let (C,D) be an ST-pair. Then there is an injective map*

$$\Psi : \mathrm{silt}\,\mathsf{C} \to t\text{-}\mathrm{str}_{bd}\mathsf{D}$$

given by $M \mapsto (\mathsf{D}_M^{\leq 0},\mathsf{D}_M^{\geq 0})$.

In the following, we give a characterisation of that Ψ is bijective from the viewpoint of silting theory. For objects M, N of T, we write $M \geq N$ if $\mathrm{Hom}_{\mathsf{T}}(M, N[n]) = 0$ for all positive integers n. Then the relation $\geq$ gives a partial order on $\mathrm{silt}\,\mathsf{T}$ by [5, Theorem 2.11]. For a basic silting object M and a positive integer n, let

$$n_M\text{-silt}\,\mathsf{T} := \{N \in \mathrm{silt}\,\mathsf{T} \mid M \geq N \geq M[n-1]\}.$$

We recall the notion of silting-discrete triangulated categories, which plays an important role in this note.

Definition 10. A triangulated category T is said to be *silting-discrete* if, for any basic silting object M, the set $n_M\text{-silt}\,\mathsf{T}$ is finite for any positive integer n.

By [4, Proposition 3.8], T is silting-discrete if and only if, for any fixed basic silting object M of T, the set $n_M\text{-silt}\,\mathsf{T}$ is finite for any positive integer n. Moreover, if T is silting-discrete, then we can obtain all basic silting objects in T from any fixed basic silting object by a finite sequence of mutations (see [4, Corollary 3.9]). By a result of [6], we have a criterion for silting-discreteness.

Lemma 11 ([6, Theorem 2.4])**.** *A triangulated category T is silting-discrete if and only if the set $2_M\text{-silt}\,\mathsf{T}$ is finite for any basic silting object M of T.*

Note that $2_M\text{-silt}\,\mathsf{T}$ corresponds bijectively to the set of isomorphism classes of basic support τ-tilting $\mathrm{End}_{\mathsf{T}}(M)$-modules (see [12] and [2]).

We collect some examples of silting-discrete triangulated categories.

Example 12. Assume that K is algebraically closed. The bounded homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ is silting-discrete if Λ is one of the following finite-dimensional K-algebras:

1. local algebras (see [5, Theorem 2.26]),
2. representation-finite hereditary algebras (see [4, Example 3.7]),
3. derived-discrete algebras of finite global dimension (see [9, Proposition 6.12]),
4. representation-finite symmetric algebras (see [4, Theorem 5.6]),
5. Brauer graph algebras whose Brauer graph contains at most one cycle of odd length and no cycle of even length (see [1, Theorem 6.7]).

We have the following theorem which is a main result of this note.

Theorem 13. *Let* (C, D) *be an ST-pair inside* T. *Then the following statements are equivalent.*

1. *The map* $\Psi : \mathrm{silt}\,\mathsf{C} \to t\text{-}\mathrm{str}_{bd}\mathsf{D}$ *is bijective.*
2. C *is silting-discrete.*
3. *The heart of every bounded* t*-structure on* D *has a projective generator.*

In the rest of this note, we give examples of silting-discrete triangulated categories by cluster tilting theory. We recall the notion of Calabi–Yau pairs. Fix an integer $d \geq 1$.

Definition 14. An ST-pair (C, D) inside C is called a $(d+1)$*-Calabi–Yau pair* if there exists a bifunctorial isomorphism for any $X \in \mathsf{D}$ and $Y \in \mathsf{C}$:

$$D\,\mathrm{Hom}_{\mathsf{C}}(X, Y) \simeq \mathrm{Hom}_{\mathsf{C}}(Y, X[d+1]).$$

If M is a silting object of C, then $(\mathsf{C}, \mathsf{D}, M)$ is a $(d+1)$*-Calabi–Yau triple* in the sense of [13, Section 5.1]. Note that, for silting objects M and N, $(\mathsf{C}, \mathsf{D}, M)$ is a $(d+1)$-Calabi–Yau triple if and only if $(\mathsf{C}, \mathsf{D}, N)$ is a $(d+1)$-Calabi–Yau triple.

Fix a $(d+1)$-Calabi–Yau pair (C, D). Consider the triangle quotient

$$\mathsf{U} := \mathsf{C}/\mathsf{D},$$

which is called the *cluster category*. Let $\pi : \mathsf{C} \to \mathsf{U}$ be the canonical projection functor. We call $T \in \mathsf{U}$ a d*-cluster tilting object* if

$$\begin{aligned} \mathsf{add}\,T &= \{X \in \mathsf{U} \mid \mathrm{Hom}_{\mathsf{U}}(X, T[i]) = 0 \text{ for } 1 \leq i \leq d-1\} \\ &= \{X \in \mathsf{U} \mid \mathrm{Hom}_{\mathsf{U}}(T, X[i]) = 0 \text{ for } 1 \leq i \leq d-1\}. \end{aligned}$$

Note that, if $d = 1$, then we have $\mathsf{add}\,T = \mathsf{U}$. We denote by $d\text{-ctilt}\,\mathsf{U}$ the set of isomorphism classes of basic d-cluster tilting objects of U. The following proposition is a basic result for Calabi-Yau triples.

Proposition 15 ([13, Theorem 5.8 and Corollary 5.12])**.** *For a* $(d+1)$*-Calabi–Yau triple* $(\mathsf{C}, \mathsf{D}, M)$*, the following statements hold.*

1. *The category* U *is a* d*-Calabi–Yau triangulated category.*
2. *The functor* π *induces an injection*

   $$d_M\text{-silt}\,\mathsf{C} \to d\text{-ctilt}\,\mathsf{U},$$

   which is a bijection if $d = 1$ *or* $d = 2$.

Now we give a criterion for C being silting-discrete in terms of the cluster category U as follows.

Theorem 16. *For a $(d+1)$-Calabi–Yau pair (C, D), the following statements hold.*

(1) *Assume $d \geq 2$. If d-$\mathrm{ctilt}\,\mathsf{U}$ is a finite set, then C is silting-discrete. The converse holds true if $d = 2$.*
(2) *Assume that $d = 1$ or 2 and let N be a basic silting object of C. Then C is silting-discrete if and only if 2_N-$\mathrm{silt}\,\mathsf{C}$ is a finite set.*

As an application of Theorem 16, we show that

- the perfect derived category of a derived preprojective algebra associated with a quiver is silting-discrete if and only if the quiver is Dynkin,
- the perfect derived category of the complete Ginzburg dg algebra associated with a quiver with a nondegenerate potential is silting-discrete if and only if the quiver is mutation equivalent to a Dynkin quiver.

Derived preprojective algebras. Let Q be a finite quiver and $d > 0$ an integer. Define a graded quiver $\tilde{Q}$ as follows: $\tilde{Q}$ has the same vertices as Q and three types of arrows

- the arrows of Q, in degree 0,
- $\alpha^* : j \to i$ in degree $-d+1$, for each arrow $\alpha : i \to j$ of Q,
- $t_i : i \to i$ in degree $-d$, for each vertex i of Q.

The *derived $(d+1)$-preprojective algebra* $\Gamma := \Gamma_{d+1}(Q)$ is the dg algebra $(K\tilde{Q}, \mathsf{d})$, where $K\tilde{Q}$ is the graded path algebra of $\tilde{Q}$ and d is the unique K-linear differential which satisfies the graded Leibniz rule

$$\mathsf{d}(ab) = \mathsf{d}(a)b + (-1)^p a\mathsf{d}(b),$$

where a is homogeneous of degree p, and which takes the following values

- $\mathsf{d}(e_i) = 0$ for any vertex i of Q, where e_i is the trivial path at i,
- $\mathsf{d}(\alpha) = 0$ for any arrow α of Q,
- $\mathsf{d}(\alpha^*) = 0$ for any arrow α^* of Q,
- $\mathsf{d}(t_i) = e_i \sum_\alpha (\alpha\alpha^* - \alpha^*\alpha)e_i$ for any vertex i of Q, where α runs over all arrows of Q.

Note that if $d = 1$, then $H^0(\Gamma)$ is the preprojective algebra associated with Q, and if $d \geq 2$, then $H^0(\Gamma)$ is the path algebra of Q.

Since Γ is concentrated in non-positive degrees, Γ is a silting object of $\mathsf{per}(\Gamma)$. Moreover, by [15, Theorem 6.3] and [16, Lemma 4.1], we have $\mathsf{D}_{\mathrm{fd}}(\Gamma) \subseteq \mathsf{per}(\Gamma)$ and there is a functorial isomorphism for $X \in \mathsf{D}_{\mathrm{fd}}(\Gamma)$ and $Y \in \mathsf{D}(\Gamma)$

$$D\,\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(Y, X[d+1]),$$

where $D := \mathrm{Hom}_K(-, K)$.

The following lemma gives an example of ST-pairs.

Lemma 17. *Let Q be a finite quiver and $\Gamma = \Gamma_{d+1}(Q)$. Then the following conditions are equivalent:*

(1) *$\mathsf{per}(\Gamma)$ is Hom-finite and Krull–Schmidt,*
(2) *$H^0(\Gamma)$ is finite-dimensional,*

(3) $d = 1$ *and* Q *is Dynkin, or* $d \geq 2$ *and* Q *has no oriented cycles.*

If these conditions are satisfied, then $(\mathsf{per}(\Gamma), \mathsf{D}_{\mathrm{fd}}(\Gamma), \Gamma)$ *is an ST-triple inside* $\mathsf{per}(\Gamma)$*, and moreover, a* $(d+1)$*-Calabi–Yau triple.*

Now we apply Theorem 16 to perfect derived categories of derived preprojective algebras.

Corollary 18. *Let* Q *be a finite quiver and* $\Gamma = \Gamma_{d+1}(Q)$*. Assume that* K *is algebraically closed and* $H^0(\Gamma)$ *is finite-dimensional. Then* $\mathsf{per}(\Gamma)$ *is silting-discrete if and only if* Q *is Dynkin.*

Complete Ginzburg dg algebras. We refer to [10] for the definition and properties of quiver mutation and mutation of quivers with potential.

Let Q be a finite quiver and W a potential. Let $\Gamma := \widehat{\Gamma}(Q, W)$ be the complete Ginzburg dg algebra associated with the quiver with potential (Q, W), see [11, 18]. The algebra $H^0\Gamma$ is known as the *Jacobian algebra*. We say that (Q, W) is *Jacobi-finite* if the Jacobian algebra is finite-dimensional.

By definition, Γ is concentrated in non-positive degrees and Γ is a silting object of $\mathsf{per}(\Gamma)$. By [18, Theorem A.16 and A.17], we obtain that (Q, W) is Jacobi-finite if and only if $(\mathsf{per}(\Gamma), \mathsf{D}_{\mathrm{fd}}(\Gamma), \Gamma)$ is a 3-Calabi–Yau triple. Now we apply Theorem 16 to perfect derived categories of complete Ginzburg dg algebras.

Corollary 19. *Let* (Q, W) *be a Jacobi-finite quiver with potential and* $\Gamma := \widehat{\Gamma}(Q, W)$*. Assume that* K *is algebraically closed and* W *is nondegenerate* (*see* [10])*. Then* $\mathsf{per}(\Gamma)$ *is silting-discrete if and only if* Q *is related to a Dynkin quiver by a finite sequence of quiver mutations.*

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