HOW TO CAPTURE \(t\)-STRUCTURES BY SILTING THEORY

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Abstract. In this note, we study a relationship between silting objects and \(t\)-structures. We introduce the notion of ST-pairs of thick subcategories of a given triangulated category, a prototypical example of which is the pair of the bounded homotopy category and the bounded derived category of a finite-dimensional algebra. For an ST-pair \((C, D)\), we construct an injective map from silting objects in \(C\) to bounded \(t\)-structures on \(D\), and show that the map is bijective if and only if \(C\) is silting-discrete. Moreover, using cluster tilting theory, we give a new class of silting-discrete triangulated categories.

This is based on a joint work with Dong Yang [3]. Throughout this note, \(K\) is a field and \(T\) is a \(K\)-linear Hom-finite Krull-Schmidt triangulated category with shift functor [1].

Our aim of this note is to give a construction of bounded \(t\)-structures by silting objects. First we recall the notion of \(t\)-structures, which was introduced by Beilinson-Bernstein-Deligne [8].

Definition 1. A \(t\)-structure on \(T\) is a pair \((T^{\leq 0}, T^{\geq 0})\) of strictly full subcategories of \(T\) such that

1. \(T^{\leq 1} \supseteq T^{\leq 0}\) and \(T^{\geq 0} \supseteq T^{\geq 1}\),
2. \(\text{Hom}_T(X, Y) = 0\) for all \(X \in T^{\leq 0}\) and \(Y \in T^{\geq 1}\),
3. for each \(Z \in T\), there is a triangle \(X \to Z \to Y \to X[1]\) in \(T\) with \(X \in T^{\leq 0}\) and \(Y \in T^{\geq 1}\).

Here, for any integer \(n\), let \(T^{\leq n} = T^{\leq 0}[-n]\) and \(T^{\geq n} = T^{\geq 0}[-n]\).

Let \((T^{\leq 0}, T^{\geq 0})\) be a \(t\)-structure on \(T\). Then the heart \(T^0 := T^{\leq 0} \cap T^{\geq 0}\) is an abelian category. We call \((T^{\leq 0}, T^{\geq 0})\) a bounded \(t\)-structure if

\[ T = \bigcup_{n \in \mathbb{Z}} T^{\leq n} = \bigcup_{n \in \mathbb{Z}} T^{\geq n}, \]

or equivalently, \(T = \text{thick} T^0\). We denote by \(t\text{-str}_{bd} T\) the set of bounded \(t\)-structures on \(T\).

We give an example of bounded \(t\)-structures.

Example 2. Let \(\Lambda\) be a finite-dimensional algebra and \(D := D^b(\text{mod}\Lambda)\) the bounded derived category. We define two full subcategories as follows:

\[ D^{\leq 0} := \{ X \in D \mid H^n X = 0 \text{ for all integers } n > 0\}, \]
\[ D^{\geq 0} := \{ X \in D \mid H^n X = 0 \text{ for all integers } n < 0\}. \]

Then it is well-known that \((D^{\leq 0}, D^{\geq 0})\) is a bounded \(t\)-structure on \(D\).

The detailed version of this note will be submitted for publication elsewhere.
Next we recall the definition of silting objects, which was introduced by Keller–Vossieck [17]. For details, we refer to [5].

**Definition 3.** An object $M$ of $\mathcal{T}$ is said to be *silting* if $\text{Hom}_\mathcal{T}(M, M[n]) = 0$ for all integers $n > 0$ and $\mathcal{T} = \text{thick} M$. We denote by $\text{silt}\mathcal{T}$ the set of isomorphism classes of basic silting objects of $\mathcal{T}$.

We give a typical example of a silting object.

**Example 4.** Let $\Lambda$ be a finite-dimensional algebra. Then $\Lambda$ is a silting object of the bounded homotopy category $K^b(\text{proj}\Lambda)$.

We introduce the notion of ST-pairs, which plays a central role in this note. For an object $M$ of $\mathcal{T}$, we define full subcategories of $\mathcal{T}$ as follows:

$$T^0_M := \{X \in \mathcal{T} | \text{Hom}_\mathcal{T}(M, X[n]) = 0 \text{ for all integers } n \neq 0\},$$

$$T^0_M := \{X \in \mathcal{T} | \text{Hom}_\mathcal{T}(M, X[n]) = 0 \text{ for all integers } n < 0\},$$

$$T^0_M := T^0_M \cap T^0_M.$$

**Definition 5.** Let $\mathcal{C}$ and $\mathcal{D}$ be thick subcategories of $\mathcal{T}$. The pair $(\mathcal{C}, \mathcal{D})$ is called an *ST-pair* inside $\mathcal{T}$ if there exists a silting object $M$ of $\mathcal{C}$ such that

(ST1) $(T^0_M, T^0_M)$ is a $t$-structure on $\mathcal{T}$,

(ST2) $T^0_M \subseteq \mathcal{D}$,

(ST3) $\mathcal{D} = \text{thick} T^0_M$.

When there is a need to emphasise the silting object $M$, we call the triple $(\mathcal{C}, \mathcal{D}, M)$ an *ST-triple*.

The following two examples are our motivating examples.

**Example 6.** Let $\Lambda$ be a finite-dimensional algebra and $\mathcal{T} := D^b(\text{mod}\Lambda)$. Then we have

$$T^0_M := \{X \in \mathcal{T} | H^n X = 0 \text{ for all integers } n > 0\},$$

$$T^0_M := \{X \in \mathcal{T} | H^n X = 0 \text{ for all integers } n < 0\}.$$

We obtain that $\Lambda$ is a silting object of $\mathcal{C} := K^b(\text{proj}\Lambda)$ and $(T^0_M, T^0_M)$ is a (bounded) $t$-structure on $\mathcal{T}$. Thus $(\mathcal{C} = K^b(\text{proj}\Lambda), \mathcal{D} := \mathcal{T} = D^b(\text{mod}\Lambda))$ is an ST-pair inside $\mathcal{T}$.

**Example 7.** Let $\Gamma$ be a $\text{dg}$ algebra satisfying the following conditions:

1. $H^n(\Gamma) = 0$ for each integer $n > 0$,
2. $H^0(\Gamma)$ is finite-dimensional,
3. $D_{\text{id}}(\Gamma) \subseteq \text{per}(\Gamma)$, where $\text{per}(\Gamma)$ is the perfect derived category of $\Gamma$ and $D_{\text{id}}(\Gamma)$ is the full subcategory of the derived category $D(\Gamma)$ consisting of $\text{dg}$ $\Gamma$-modules whose total cohomology is finite-dimensional.

Let $\mathcal{T} := \text{per}(\Gamma)$. Then $\mathcal{T}$ is Hom-finite Krull–Schmidt by [14, Proposition 2.5], $(T^0_M, T^0_M)$ is a $t$-structure on $\mathcal{T}$ and $T^0_M \subseteq \text{thick} T^0_M$ (see [7, Proposition 2.7] and [14, Propositions 2.5 and 2.1(c)]). Since $\Gamma$ is a non-positive $\text{dg}$ algebra, $\Gamma$ is a silting object of $\mathcal{T}$. Moreover, we have $D_{\text{id}}(\Gamma) = \text{thick} T^0_M$. Thus $(\mathcal{C} := \mathcal{T} = \text{per}(\Gamma), \mathcal{D} := D_{\text{id}}(\Gamma))$ is an ST-pair inside $\mathcal{T}$.
Fix an ST-pair \((C, D)\). For a silting object \(M\) of \(C\), we define full subcategories of \(D\) as follows:

\[
D^\leq_M := T^\leq_M \cap D = \{X \in D \mid \text{Hom}_T(M, X[n]) = 0 \text{ for all integers } n > 0\},
\]

\[
D^\geq_M := T^\geq_M \cap D = T^\geq_M,
\]

\[
D^0_M := D^\leq_M \cap D^\geq_M.
\]

The following proposition implies that the conditions (ST1–3) are satisfied for all silting objects of \(C\), which allows us to define a well-defined map from silting objects in \(C\) to bounded \(t\)-structures on \(D\).

**Proposition 8.** Let \((C, D, M)\) be an ST-triple and let \(N\) be an arbitrary silting object of \(C\). Then the following statements hold.

1. \((C, D, N)\) is an ST-triple.
2. \(T^0_N \simeq \mod \text{End}_T(N)\).
3. \((D^\leq_N, D^\geq_N)\) is a bounded \(t\)-structure on \(D\) and \(D^0_N = T^0_N\).

The following theorem is one of our main results in this note.

**Theorem 9.** Let \((C, D)\) be an ST-pair. Then there is an injective map \(\Psi : \text{silt } C \to t\text{-str}_{bd} D\) given by \(M \mapsto (D^\leq_M, D^\geq_M)\).

In the following, we give a characterisation of that \(\Psi\) is bijective from the viewpoint of silting theory. For objects \(M, N\) of \(T\), we write \(M \geq N\) if \(\text{Hom}_T(M, N[n]) = 0\) for all positive integers \(n\). Then the relation \(\geq\) gives a partial order on \(\text{silt } T\) by [5, Theorem 2.11]. For a basic silting object \(M\) and a positive integer \(n\), let

\[n_M\text{-silt } T := \{N \in \text{silt } T \mid M \geq N \geq M[n-1]\}.
\]

We recall the notion of silting-discrete triangulated categories, which plays an important role in this note.

**Definition 10.** A triangulated category \(T\) is said to be silting-discrete if, for any basic silting object \(M\), the set \(n_M\text{-silt } T\) is finite for any positive integer \(n\).

By [4, Proposition 3.8], \(T\) is silting-discrete if and only if, for any fixed basic silting object \(M\) of \(T\), the set \(n_M\text{-silt } T\) is finite for any positive integer \(n\). Moreover, if \(T\) is silting-discrete, then we can obtain all basic silting objects in \(T\) from any fixed basic silting object by a finite sequence of mutations (see [4, Corollary 3.9]). By a result of [6], we have a criterion for silting-discreteness.

**Lemma 11** ([6, Theorem 2.4]). A triangulated category \(T\) is silting-discrete if and only if the set \(2_M\text{-silt } T\) is finite for any basic silting object \(M\) of \(T\).

Note that \(2_M\text{-silt } T\) corresponds bijectively to the set of isomorphism classes of basic support \(\tau\)-tilting \(\text{End}_T(M)\)-modules (see [12] and [2]).

We collect some examples of silting-discrete triangulated categories.

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Example 12. Assume that $K$ is algebraically closed. The bounded homotopy category $K^b(\text{proj}\Lambda)$ is silting-discrete if $\Lambda$ is one of the following finite-dimensional $K$-algebras:

1. local algebras (see [5, Theorem 2.26]),
2. representation-finite hereditary algebras (see [4, Example 3.7]),
3. derived-discrete algebras of finite global dimension (see [9, Proposition 6.12]),
4. representation-finite symmetric algebras (see [4, Theorem 5.6]),
5. Brauer graph algebras whose Brauer graph contains at most one cycle of odd length and no cycle of even length (see [1, Theorem 6.7]).

We have the following theorem which is a main result of this note.

**Theorem 13.** Let $(C, D)$ be an ST-pair inside $T$. Then the following statements are equivalent.

1. The map $\Psi : \text{silt} C \to \text{t-str}_{td} D$ is bijective.
2. $C$ is silting-discrete.
3. The heart of every bounded $t$-structure on $D$ has a projective generator.

In the rest of this note, we give examples of silting-discrete triangulated categories by cluster tilting theory. We recall the notion of Calabi-Yau pairs. Fix an integer $d \geq 1$.

**Definition 14.** An ST-pair $(C, D)$ inside $C$ is called a $(d + 1)$-Calabi-Yau pair if there exists a bifunctorial isomorphism for any $X \in D$ and $Y \in C$:

$$D \text{Hom}_C(X, Y) \simeq \text{Hom}_C(Y, X[d + 1]).$$

If $M$ is a silting object of $C$, then $(C, D, M)$ is a $(d + 1)$-Calabi-Yau triple in the sense of [13, Section 5.1]. Note that, for silting objects $M$ and $N$, $(C, D, M)$ is a $(d + 1)$-Calabi-Yau triple if and only if $(C, D, N)$ is a $(d + 1)$-Calabi-Yau triple.

Fix a $(d + 1)$-Calabi-Yau pair $(C, D)$. Consider the triangle quotient

$$U := C/D,$$

which is called the cluster category. Let $\pi : C \to U$ be the canonical projection functor. We call $T \in U$ a $d$-cluster tilting object if

$$\text{add} T = \{X \in U \mid \text{Hom}_U(X, T[i]) = 0 \text{ for } 1 \leq i \leq d - 1\} = \{X \in U \mid \text{Hom}_U(T, X[i]) = 0 \text{ for } 1 \leq i \leq d - 1\}.$$

Note that, if $d = 1$, then we have $\text{add} T = U$. We denote by $d$-ctilt $U$ the set of isomorphism classes of basic $d$-cluster tilting objects of $U$. The following proposition is a basic result for Calabi-Yau triples.

**Proposition 15** ([13, Theorem 5.8 and Corollary 5.12]). For a $(d + 1)$-Calabi-Yau triple $(C, D, M)$, the following statements hold.

1. The category $U$ is a $d$-Calabi-Yau triangulated category.
2. The functor $\pi$ induces an injection

$$d_M \text{-silt } C \to d \text{-ctilt } U,$$

which is a bijection if $d = 1$ or $d = 2$. 

Now we give a criterion for \( C \) being silting-discrete in terms of the cluster category \( U \) as follows.

**Theorem 16.** For a \((d+1)-\text{Calabi–Yau pair} \ (C,D)\), the following statements hold.

1. Assume \( d \geq 2 \). If \( d\text{-ctilt} \ U \) is a finite set, then \( C \) is silting-discrete. The converse holds true if \( d = 2 \).
2. Assume that \( d = 1 \) or \( 2 \) and let \( N \) be a basic silting object of \( C \). Then \( C \) is silting-discrete if and only if \( 2N\text{-silt} \ C \) is a finite set.

As an application of Theorem 16, we show that

- the perfect derived category of a derived preprojective algebra associated with a quiver is silting-discrete if and only if the quiver is Dynkin,
- the perfect derived category of the complete Ginzburg dg algebra associated with a quiver with a nondegenerate potential is silting-discrete if and only if the quiver is mutation equivalent to a Dynkin quiver.

**Derived preprojective algebras.** Let \( Q \) be a finite quiver and \( d > 0 \) an integer. Define a graded quiver \( \widetilde{Q} \) as follows: \( \widetilde{Q} \) has the same vertices as \( Q \) and three types of arrows

- the arrows of \( Q \), in degree 0,
- \( \alpha^* : j \rightarrow i \) in degree \(-d+1\), for each arrow \( \alpha : i \rightarrow j \) of \( Q \),
- \( t_i : i \rightarrow i \) in degree \(-d\), for each vertex \( i \) of \( Q \).

The \textit{derived \((d+1)\)-preprojective algebra} \( \Gamma := \Gamma_{d+1}(Q) \) is the dg algebra \((K\widetilde{Q},d)\), where \( K\widetilde{Q} \) is the graded path algebra of \( \widetilde{Q} \) and \( d \) is the unique \( K \)-linear differential which satisfies the graded Leibniz rule

\[
\text{d}(ab) = \text{d}(a)b + (-1)^p a\text{d}(b),
\]

where \( a \) is homogeneous of degree \( p \), and which takes the following values

- \( \text{d}(e_i) = 0 \) for any vertex \( i \) of \( Q \), where \( e_i \) is the trivial path at \( i \),
- \( \text{d}(\alpha) = 0 \) for any arrow \( \alpha \) of \( Q \),
- \( \text{d}(\alpha^*) = 0 \) for any arrow \( \alpha^* \) of \( Q \),
- \( \text{d}(t_i) = e_i \sum_{\alpha}(\alpha\alpha^* - \alpha^*\alpha)e_i \) for any vertex \( i \) of \( Q \), where \( \alpha \) runs over all arrows of \( Q \).

Note that if \( d = 1 \), then \( H^0(\Gamma) \) is the preprojective algebra associated with \( Q \), and if \( d \geq 2 \), then \( H^0(\Gamma) \) is the path algebra of \( Q \).

Since \( \Gamma \) is concentrated in non-positive degrees, \( \Gamma \) is a silting object of \( \text{per}(\Gamma) \). Moreover, by [15, Theorem 6.3] and [16, Lemma 4.1], we have \( D_\text{fd}(\Gamma) \subseteq \text{per}(\Gamma) \) and there is a functorial isomorphism for \( X \in D_\text{fd}(\Gamma) \) and \( Y \in D(\Gamma) \)

\[
D \text{Hom}(X,Y) \simeq \text{Hom}(Y,X[d+1]),
\]

where \( D := \text{Hom}_K(-,K) \).

The following lemma gives an example of ST-pairs.

**Lemma 17.** Let \( Q \) be a finite quiver and \( \Gamma = \Gamma_{d+1}(Q) \). Then the following conditions are equivalent:

1. \( \text{per}(\Gamma) \) is Hom-finite and Krull–Schmidt,
2. \( H^0(\Gamma) \) is finite-dimensional,
(3) \( d = 1 \) and \( Q \) is Dynkin, or \( d \geq 2 \) and \( Q \) has no oriented cycles.

If these conditions are satisfied, then \((\text{per}(\Gamma), \text{D}_{\text{rd}}(\Gamma), \Gamma)\) is an ST-triple inside \( \text{per}(\Gamma) \), and moreover, a \((d + 1)\)-Calabi–Yau triple.

Now we apply Theorem 16 to perfect derived categories of derived preprojective algebras.

**Corollary 18.** Let \( Q \) be a finite quiver and \( \Gamma = \Gamma_{d+1}(Q) \). Assume that \( K \) is algebraically closed and \( H^0(\Gamma) \) is finite-dimensional. Then \( \text{per}(\Gamma) \) is silting-discrete if and only if \( Q \) is Dynkin.

**Complete Ginzburg dg algebras.** We refer to [10] for the definition and properties of quiver mutation and mutation of quivers with potential.

Let \( Q \) be a finite quiver and \( W \) a potential. Let \( \Gamma := \hat{\Gamma}(Q, W) \) be the complete Ginzburg dg algebra associated with the quiver with potential \((Q, W)\), see [11, 18]. The algebra \( H^0\Gamma \) is known as the Jacobian algebra. We say that \((Q, W)\) is Jacobi-finite if the Jacobian algebra is finite-dimensional.

By definition, \( \Gamma \) is concentrated in non-positive degrees and \( \Gamma \) is a silting object of \( \text{per}(\Gamma) \). By [18, Theorem A.16 and A.17], we obtain that \((Q, W)\) is Jacobi-finite if and only if \((\text{per}(\Gamma), \text{D}_{\text{rd}}(\Gamma), \Gamma)\) is a 3-Calabi–Yau triple. Now we apply Theorem 16 to perfect derived categories of complete Ginzburg dg algebras.

**Corollary 19.** Let \((Q, W)\) be a Jacobi-finite quiver with potential and \( \Gamma := \hat{\Gamma}(Q, W) \). Assume that \( K \) is algebraically closed and \( W \) is nondegenerate (see [10]). Then \( \text{per}(\Gamma) \) is silting-discrete if and only if \( Q \) is related to a Dynkin quiver by a finite sequence of quiver mutations.

**References**


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—7—