SINGULARITY CATEGORIES AND SILTING OBJECTS

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Abstract. In this note, we discuss the existence of silting objects in triangulated categories.

Introduction

Tilting theory is now an essential tool in the study of finite dimensional algebras, and it influences many branches of mathematics. In the theory, silting objects play a central and important role. Then, we would think how many such objects there exist.

Throughout this note, Λ denotes a finite dimensional algebra over a field k.

The perfect derived category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$ of Λ has a trivial silting (tilting) object Λ, and silting mutation makes infinitely many silting objects in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Moreover, the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ over Λ has a silting object if and only if Λ is of finite global dimension. On the other hand, we know that the stable module category of a non-semisimple selfinjective algebra admits no silting object. (See [1].)

These facts inspire us with the idea that no non-zero silting object belongs to the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of Λ, which is the Verdier quotient of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ by $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Indeed, if the global dimension of Λ is finite, then the singularity category is zero. If Λ is (non-semisimple) selfinjective, then the stable module category is triangle equivalent to the singularity category [5]. In both the cases, the singularity categories never have a non-zero silting object.

A main result of this note is the following:

Theorem 1. *The singularity category of Λ admits no non-zero silting object if Λ has finite right selfinjective dimension.*

1. Silting theory

1.1. **Silting objects.** Throughout this note, let $\mathcal{T}$ be a Krull-Schmidt triangulated category which is k-linear and Hom-finite .

Let us recall the definition of silting objects.

Definition 2. An object T of $\mathcal{T}$ is said to be *presilting* if it satisfies $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any positive integer $i > 0$. It is called *silting* if in additional $\mathcal{T} = \mathsf{thick}\,T$. Here, $\mathsf{thick}\,T$ stands for the smallest thick subcategory of $\mathcal{T}$ containing T.

We denote by $\mathsf{silt}\,\mathcal{T}$ the set of isomorphism classes of basic silting objects of $\mathcal{T}$.

A typical example of silting objects is the stalk complex Λ (and its shifts) in the perfect derived category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$. Moreover, we easily observe the following.

The detailed version of this paper will be submitted for publication elsewhere.

Example 3. Let Λ be an algebra presented by the quiver $1 \to 2$. It is well-known that the bounded derived category over Λ admits the Auslander-Reiten quiver of the form:

$$\begin{tikzcd}
\cdots \arrow[dr] & & 0 \arrow[dr] & & 2 \arrow[dr] & \\
& -1 \arrow[ur] & & 1 \arrow[ur] & & \cdots ,
\end{tikzcd}$$

where the shift [1] is given by the operation '+3'. (See [3].) Then, the following equalities are obtained easily:

$$\mathsf{silt}\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \quad = \{i \oplus j \mid 0 < j - i \equiv 1 \pmod 3\}.$$

We already know the following triangulated categories without a silting object.

Example 4. [1, Example 2.5]

(1) The bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ has a silting object if and only if Λ is of finite global dimension. In particular, if Λ has infinite global dimension, then the bounded derived category does not contain a silting object.

(2) Let Λ be a non-semisimple selfinjective algebra. Then no silting object belongs to the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of Λ.

Our main theorem gives a slight generalization of Example 4 (2).

1.2. **Silting reduction.** To prove Theorem 1, silting reduction plays a crucial role, which had been introduced in [1] and was developed in [4].

In this subsection, fix a presilting object T of $\mathcal{T}$. We define a subset $\mathsf{silt}_T\,\mathcal{T}$ of $\mathsf{silt}\,\mathcal{T}$ by

$$\mathsf{silt}_T\,\mathcal{T} := \{P \in \mathsf{silt}\,\mathcal{T} \mid T \text{ is a direct summand of } P\}.$$

Moreover, one puts $\mathcal{S} := \mathsf{thick}\,T$. The Verdier quotient of $\mathcal{T}$ by $\mathcal{S}$ is written as $\mathcal{T}/\mathcal{S}$.

Then, silting reduction [4, Theorem 3.7] says:

Theorem 5. *The canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ gives rise to a bijection $\mathsf{silt}_T\,\mathcal{T} \to \mathsf{silt}\,\mathcal{T}/\mathcal{S}$ if any object X of $\mathcal{T}$ satisfies $\mathrm{Hom}_{\mathcal{T}}(X, T[\ell]) = 0 = \mathrm{Hom}_{\mathcal{T}}(T, X[\ell])$ for $\ell \gg 0$.*

The assumption of this theorem is satisfied when $\mathcal{T}$ contains a silting object; see [1, Proposition 2.4] and for example consider $\mathcal{T} = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$.

2. Proof and corollary of Theorem 1

We are now ready to show our main theorem.

Proof of Theorem 1. We will apply silting reduction to $\mathcal{T} = \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda)$ and $T = \Lambda$.

We check that the conditions $\mathrm{Hom}_{\mathcal{T}}(X, \Lambda[\ell]) = 0 = \mathrm{Hom}_{\mathcal{T}}(\Lambda, X[\ell])$ are satisfied. The second equality holds evidently. We obtain also the first equality, since the injective dimension of Λ_Λ is finite by our assumption.

Now, we apply silting reduction. As $\mathcal{S} = \mathsf{thick}\,\Lambda = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,\Lambda)$, it is seen that the Verdier quotient $\mathcal{T}/\mathcal{S}$ is just the singularity category $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ of Λ. By Theorem 5, we have a bijection $\mathsf{silt}_\Lambda\,\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,\Lambda) \to \mathsf{silt}\,\mathsf{D}_{\mathsf{sg}}(\Lambda)$, whence it follows from Example 4 (1) that $\mathsf{D}_{\mathsf{sg}}(\Lambda)$ has no non-zero silting object. $\square$

We say that Λ is *Iwanaga-Gorenstein* if it has finite right and left selfinjective dimension. Then, as is well-known, the full subcategory

$$\mathsf{CM}\,\Lambda := \{M \in \mathsf{mod}\,\Lambda \mid \operatorname{Ext}^i_\Lambda(M,\Lambda) = 0 \text{ for any } i > 0\}$$

of $\mathsf{mod}\,\Lambda$ is Frobenius, and hense its stable category $\underline{\mathsf{CM}}\,\Lambda$ is triangulated [3]. Thanks to Buchweitz's theorem [2], we have a triangle equivalence $\mathsf{D}_{\mathsf{sg}}(\Lambda) \to \underline{\mathsf{CM}}\,\Lambda$.

Thus, the following corollary is an immediate consequence of Theorem 1.

Corollary 6. *The stable category $\underline{\mathsf{CM}}\,\Lambda$ has no non-zero silting object if Λ is Iwanaga-Gorenstein.*

References

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