SILTING COMPLEXES OVER RADICAL SQUARE ZERO ALGEBRAS

TOSHITAKA AOKI

Abstract. In this note, we give an explicit description of two-term silting complexes over algebras with radical square zero. As an application, we will count the number of two-term tilting complexes over Brauer line algebras up to isomorphisms.

1. Preliminaries

Throughout this note, by an algebra we mean a finite dimensional basic algebra over an algebraically closed field k. For an algebra Λ, we denote by $\mathrm{proj}\Lambda$ the category of finitely generated projective right Λ-modules, by $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ the homotopy category of bounded complexes of $\mathrm{proj}\Lambda$. An object X is said to be basic if it is isomorphic to a direct sum of indecomposable objects which are mutually non-isomorphic. We denote by $\mathsf{add}X$ the category of all direct summands of direct sums of copies of X.

In this section, we study basic properties of silting complexes, in particular, we are interested in two-term silting complexes. Let Λ be an algebra.

Definition 1. Let T be a complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.

(1) We say that T is *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(T, T[n]) = 0$ for all positive integers n.
(2) We say that T is *silting* if it is presilting and generates $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ by taking direct sums, direct summands, shifts and mapping cones. In addition, it is said to be *tilting* if it also satisfies $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(T, T[n]) = 0$ for all non-zero integers n.
(3) We say that T is *two-term* if it is isomorphic to a complex concentrated in degree 0 and -1 in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$, i.e., of the form $(0 \to T^{-1} \to T^0 \to 0)$.

We denote by $\mathsf{2\text{-}silt}\Lambda$ (respectively, $\mathsf{2\text{-}tilt}\Lambda$) the set of isomorphism classes of basic two-term silting (respectively, tilting) complexes for Λ. For example, any tilting module over Λ provides a two-term tilting complex, where a finitely generated right Λ-module M is said to be *tilting* if it satisfies the following conditions: (i) The projective dimension of M is at most 1, (ii) $\mathrm{Ext}^1_\Lambda(M, M) = 0$ and (iii) there exists a short exact sequence $0 \to \Lambda \to M' \to M'' \to 0$ of right Λ-modules with M', M'' in $\mathsf{add}M$. We denote by $\mathsf{tilt}\Lambda$ the set of isomorphism classes of basic tilting modules over Λ. It is also known that they have the structure of partially ordered sets, which comes from silting theory in a more general setting, defined by $T \geq U$ whenever $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(T, U[n]) = 0$ for all positive integers n.

We need the following observation for two-term complexes.

Proposition 2. [3] *Let $T = (T^{-1} \to T^0)$ be a two-term presilting complex in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$. Then the following hold:*

The detailed version of this paper will be submitted for publication elsewhere.

(1) *T is silting if and only if $|T| = |\Lambda|$, where $|X|$ is the number of isomorphism classes of indecomposable direct summands of X in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.*
(2) $\mathsf{add}T^0 \cap \mathsf{add}T^{-1} = 0$ *holds.*
(3) *If T is silting, then $\Lambda \in \mathsf{add}(T^0 \oplus T^{-1})$ holds.*

Let $Q = (Q_0, Q_1)$ be the quiver of Λ, where Q_0 is the vertex set and Q_1 is the arrow set. Suppose $\epsilon\colon Q_0 \to \{+, -\}$ is a map, called a *signature* on Q, and $Q_0^+ := \{i \in Q_0 \mid \epsilon(i) = +\}$, $Q_0^- := \{j \in Q_0 \mid \epsilon(j) = -\}$. We denote by $\mathsf{K}_\epsilon^2(\mathrm{proj}\Lambda)$ a full subcategory of $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ consisting of all two-term complexes $T \cong (T^{-1} \to T^0)$ such that $T^0 \in \mathsf{add}\left(\bigoplus_{i\in Q_0^+} P_i\right)$ and $T^{-1} \in \mathsf{add}\left(\bigoplus_{j\in Q_0^-} P_j\right)$, where P_i is the indecomposable projective right Λ-module concentrated in $i \in Q_0$.

Remark 3. The category $\mathsf{K}_\epsilon^2(\mathrm{proj}\Lambda)$ is closed under extensions. In other words, if we have a triangle $U \to V \to T \to U[1]$ with $T, U \in \mathsf{K}_\epsilon^2(\mathrm{proj}\Lambda)$, then V is also in $\mathsf{K}_\epsilon^2(\mathrm{proj}\Lambda)$. It is obvious from the definition of a mapping cone.

The importance of this category in silting theory is highlighted in the following result, which is immediately from Proposition 2 (2) and (3). Now, we denote by $\mathsf{2\text{-}silt}_\epsilon\Lambda$ a subset of $\mathsf{2\text{-}silt}\Lambda$ consisting of all complexes which lie in $\mathsf{K}_\epsilon^2(\mathrm{proj}\Lambda)$.

Proposition 4. [5] *The set $\mathsf{2\text{-}silt}\Lambda$ is decomposed into a disjoint union of sets: $\mathsf{2\text{-}silt}\Lambda = \bigcup_\epsilon \mathsf{2\text{-}silt}_\epsilon\Lambda$, where ϵ runs over all signatures on Q.*

The above discussion suggests that we may just focus on $\mathsf{2\text{-}silt}_\epsilon\Lambda$ on the category $\mathsf{K}_\epsilon^2(\mathrm{proj}\Lambda)$ for each signature ϵ to obtain all two-term silting complexes for Λ.

2. Main Results

Our aim of this note is to study two-term silting complexes for the algebras with radical square zero, that is, any such an algebra Λ satisfies $J_\Lambda^2 = 0$, where J_Λ is the Jacobson radical of Λ. Let Λ be an algebra with radical square zero and $Q = (Q_0, Q_1)$ the quiver of Λ, then Λ is presented as a path algebra of Q modulo an ideal generated by all paths of length 2. For each signature ϵ on Q, we define Q_ϵ as a subquiver of Q with vertex set Q_0 and arrow set

$$\{i \to j \text{ in } Q_1 \mid i \in Q_0^+ \text{ and } j \in Q_0^-\}.$$

Note that Q_ϵ is bipartite, that is, every vertex in Q_ϵ is either a source or a sink. We denote by Q_ϵ^{op} its opposite quiver.

The main result in this note is the following strengthened result of Proposition 4.

Theorem 5. [5] *For a signature ϵ on Q, we have bijections between the following sets:*
(1) $\mathsf{2\text{-}silt}_\epsilon\Lambda$,
(2) $\mathsf{2\text{-}silt}_\epsilon(kQ_\epsilon)$,
(3) $\mathsf{tilt}(kQ_\epsilon^{\mathrm{op}})$.

Moreover, they induce isomorphisms of partially ordered sets.

We will describe these bijections in Example 7. On the other hand, we are also interested in knowing how many two-term silting complexes exist for a given algebra Λ (up to isomorphisms). From the point of view of τ-tilting theory, the algebra Λ is said to be *τ-tilting-finite* if the set $\mathsf{2\text{-}silt}\Lambda$ is finite. The first part of the following result is also obtained by Adachi [2].

Corollary 6. [5] *Let Λ be an algebra with radical square zero and Q the quiver of Λ. Then Λ is τ-tilting-finite if and only if Q_ϵ is a disjoint union of Dynkin quivers for every signature ϵ on Q. Moreover, in this case, we can calculate the cardinality $\#\mathsf{2\text{-}silt}\Lambda$ by using numbers $\#\mathsf{tilt}(k\Delta)$ for Dynkin quivers Δ appearing in Q_ϵ with:*

Δ	$\mathbb{A}_n$	$\mathbb{D}_n\,(n \geq 4)$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\#\mathsf{tilt}(k\Delta)$	$\frac{1}{n+1}\binom{2n}{n}$	$\frac{3n-4}{2n-2}\binom{2n-2}{n-1}$	418	2431	17342

Example 7. Let Q be the following quiver:

$$1 \longrightarrow 2 \rightleftarrows 3.$$

Then the algebra $\Lambda := kQ/I$ is an algebra with radical square zero where I is the ideal in kQ generated by all paths of length 2. By Corollary 6, it is τ-tilting-finite since every component of Q_ϵ associated with ϵ is of type $\mathbb{A}_n$ for any signature ϵ on Q. Table 1 describes the correspondence of objects in Theorem 5. Here, we will impose the following: For a given signature ϵ, the vertex i with $\epsilon(i) = \sigma$ is written as i^σ instead for simplicity, and modules over $kQ_\epsilon^{\mathrm{op}}$ are written as representations of a quiver. And we also suppose that each (non-zero) morphism between modules is given by a natural one. Then the object in a cell corresponds to a tilting module/silting complex by taking a direct sum of (three) indecomposable objects in this cell.

A trivial example of this correspondence is the case when $Q_\epsilon^{\mathrm{op}} = (1^+\,2^+\,3^+)$. The algebra $kQ_\epsilon^{\mathrm{op}}$ is semisimple and has exactly one tilting module, namely $kQ_\epsilon^{\mathrm{op}}$ itself. And it corresponds to a stalk complex Λ in $\mathsf{2\text{-}silt}_\epsilon\Lambda$. It is easy to see that there are $2^3 = 8$ choices of signatures on Q, and consequently we have $\#\mathsf{2\text{-}silt}\Lambda = 16$.

According to the following result, the silting theory of symmetric radical cube zero algebras is closely related to that of radical square zero algebras. Note that any silting complex over a symmetric algebra is tilting [3].

Lemma 8. [1] *Let Λ be an indecomposable symmetric radical cube zero algebra with $J_\Lambda^2 \neq 0$, then $\overline{\Lambda} := \Lambda/\mathrm{soc}\Lambda$ is radial square zero, where $\mathrm{soc}\Lambda$ is the socle of Λ. In this situation, we have an isomorphism of partially ordered sets:*

$$\mathsf{2\text{-}tilt}\Lambda \longrightarrow \mathsf{2\text{-}silt}\overline{\Lambda}.$$

This fact provides many examples of symmetric algebras whose two-term tilting complexes are computable. In particular, there is an important class of such algebras called *Brauer line algebras*, which is also the special class of Brauer tree algebras (we refer to [4]). We denote by Γ_n the Brauer tree algebra corresponding to a (multiplicity-free) Brauer tree of the line with n edges, i.e., of the form:

$$\bullet \overset{1}{\longrightarrow} \bullet \overset{2}{\longrightarrow} \bullet \cdots \bullet \overset{n}{\longrightarrow} \bullet$$

Then Γ_n is a representation-finite symmetric algebra with radical cube zero. As a consequence, we obtain the following formula for calculating the number of two-term tilting complexes over Brauer line algebras.

Q_ϵ^{op}	$\mathsf{tilt}(kQ_\epsilon^{\mathrm{op}})$			$\mathsf{2\text{-}silt}_\epsilon\Lambda$		
$1^+ \quad 2^+ \quad 3^+$	$(k \quad 0 \quad 0)$ $(0 \quad k \quad 0)$ $(0 \quad 0 \quad k)$			$(0 \to P_1)$ $(0 \to P_2)$ $(0 \to P_3)$		
$1^+ \quad 2^+ \leftarrow 3^-$	$(k \quad 0 \leftarrow 0)$ $(0 \quad k \leftarrow k)$ $(0 \quad k \leftarrow 0)$	$(k \quad 0 \leftarrow 0)$ $(0 \quad k \leftarrow k)$ $(0 \quad 0 \leftarrow k)$		$(0 \to P_1)$ $(P_3 \to P_2)$ $(0 \to P_2)$	$(0 \to P_1)$ $(P_3 \to P_2)$ $(P_3 \to 0)$	
$1^+ \leftarrow 2^- \to 3^+$	$(k \leftarrow 0 \to 0)$ $(k \leftarrow k \to k)$ $(0 \leftarrow 0 \to k)$	$(0 \leftarrow k \to k)$ $(k \leftarrow k \to k)$ $(0 \leftarrow 0 \to k)$	$(0 \leftarrow k \to k)$ $(0 \leftarrow k \to 0)$ $(k \leftarrow k \to 0)$	$(0 \to P_1)$ $(P_2 \to P_1 \oplus P_3)$ $(0 \to P_3)$	$(P_2 \to P_3)$ $(P_2 \to P_1 \oplus P_3)$ $(0 \to P_3)$	$(P_2 \to P_3)$ $(P_2 \to 0)$ $(P_2 \to P_1)$
	$(k \leftarrow 0 \to 0)$ $(k \leftarrow k \to k)$ $(k \leftarrow k \to 0)$	$(0 \leftarrow k \to k)$ $(k \leftarrow k \to k)$ $(k \leftarrow k \to 0)$		$(0 \to P_1)$ $(P_2 \to P_1 \oplus P_3)$ $(P_2 \to P_1)$	$(P_2 \to P_3)$ $(P_2 \to P_1 \oplus P_3)$ $(P_2 \to P_1)$	
$1^+ \leftarrow 2^- \quad 3^-$	$(k \leftarrow 0 \quad 0)$ $(k \leftarrow k \quad 0)$ $(0 \leftarrow 0 \quad k)$	$(0 \leftarrow k \quad 0)$ $(k \leftarrow k \quad 0)$ $(0 \leftarrow 0 \quad k)$		$(0 \to P_1)$ $(P_2 \to P_1)$ $(P_3 \to 0)$	$(P_2 \to 0)$ $(P_2 \to P_1)$ $(P_3 \to 0)$	
$1^- \quad 2^+ \quad 3^+$	$(k \quad 0 \quad 0)$ $(0 \quad k \quad 0)$ $(0 \quad 0 \quad k)$			$(P_1 \to 0)$ $(0 \to P_2)$ $(0 \to P_3)$		
$1^- \quad 2^+ \leftarrow 3^-$	$(k \quad 0 \leftarrow 0)$ $(0 \quad k \leftarrow 0)$ $(0 \quad k \leftarrow k)$	$(k \quad 0 \leftarrow 0)$ $(0 \quad 0 \leftarrow k)$ $(0 \quad k \leftarrow k)$		$(P_1 \to 0)$ $(0 \to P_2)$ $(P_3 \to P_2)$	$(P_1 \to 0)$ $(P_3 \to 0)$ $(P_3 \to P_2)$	
$1^- \quad 2^- \to 3^+$	$(k \quad 0 \to 0)$ $(0 \quad k \to k)$ $(0 \quad 0 \to k)$	$(k \quad 0 \to 0)$ $(0 \quad k \to k)$ $(0 \quad k \to 0)$		$(P_1 \to 0)$ $(P_2 \to P_3)$ $(0 \to P_3)$	$(P_1 \to 0)$ $(P_2 \to P_3)$ $(P_2 \to 0)$	
$1^- \quad 2^- \quad 3^-$	$(k \quad 0 \quad 0)$ $(0 \quad k \quad 0)$ $(0 \quad 0 \quad k)$			$(P_1 \to 0)$ $(P_2 \to 0)$ $(P_3 \to 0)$		

Table 1.

Theorem 9. *In the above notation, the following equation holds:*

$$\#\mathsf{2\text{-}tilt}\Gamma_n = \binom{2n}{n}.$$

References

[1] T. Adachi, *The classification of τ-tilting modules over Nakayama algebras*, J. Algebra 452 (2016), 227-262.

[2] T. Adachi, *Characterizing τ-tilting finite algebras with radical square zero*, Proc. Amer. Math. Soc. 144 (2016), no. 11, 4673-4685.

[3] T. Aihara, *Tilting-connected symmetric algebras*, Algebr. Represent. Theory 16 (2013), no. 3, 873-894.

[4] J. L. Alpelin, *Local Representation Theory*, Cambridge Studies in Advanced Mathematics 11, Cambridge University Press, Cambridge, 1986.

[5] T. Aoki, *Two-term silting complexes over radical square zero algebras*, in preparation.

Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 JAPAN
E-mail address: m15001d@math.nagoya-u.ac.jp