TWO CONSTRUCTIONS ON IWANAGA-GORENSTEIN ALGEBRAS

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Abstract. We consider two series $(A^{(m)})_{m\geq 1}$, $(A^{[m]})_{m\geq 1}$ of algebras constructed from a given algebra A via two different methods. It is already known that in the case when A is hereditary, the global dimensions of the algebras in both series form an increasing sequence. Moreover, the same holds for the series of dominant dimensions of $A^{[m]}$. We refine this result by determining the precise global and dominant dimensions of the algebras in both series to a more general subclass of Iwanaga-Gorenstein algebras. In particular, we show conditions for which higher Auslander algebras and minimal Auslander-Gorenstein algebras appear in these series.

Key Words: dominant dimension, Auslander algebras, Iwanaga-Gorenstein algebras.

This is a report on some of the results presented in [2], which is a joint work with Osamu Iyama and René Marczinzik. Throughout, we work over a field $\Bbbk$. For simplicity, by an algebra we always mean a finite dimensional $\Bbbk$-algebra that is basic, ring-indecomposable, and non-simple.

1. Two series of algebras arising from a hereditary algebra

Let us start by recalling that the *dominant dimension* of an algebra A. Suppose $\{I^n\}_{n\geq 0}$ is the minimal injective coresolution of A, then the dominant dimension of A is the infinmum n so that I^n is non-projective. In particular, the dominant dimension of a self-injective algebra is infinite. One interesting feature of dominant dimension is the so-called Morita-Tachikawa correspondence, which gives the following double centraliser property: any algebra of dominant dimension at least 2 comes with a unique idempotent so that the algebra and its idempotent truncation are centralisers of each other. A particular example of this is the Schur algebra (of an appropriate choice of parameter), where the corresponding idempotent truncation is the group algebra of symmetric group. In particular, Morita-Tachikawa correspondence can be regarded as an abstraction of an appropriate form of the Schur-Weyl duality.

In the representation theory of finite dimensional algebras, dominant dimension appears in one of the equivalent conditions for an algebra to be (higher) Auslander. Namely, an algebra is a *higher Auslander algebra* if its dominant dimension coincide with its global dimension. For example, the classical Auslander algebra is the case when this dimension is 2.

Our starting point is an article of Kunio Yamagata [3] on Frobenius algebras, where the following construction was introduced: For a given finite dimensional algebra $A = A^{[0]}$,

The detailed version of this paper will be submitted for publication elsewhere.

we define a series of finite dimensional algebras by

$$A^{[m+1]} := \mathrm{End}_{A^{[m]}}(A^{[m]} \oplus DA^{[m]}).$$

We call $A^{[m]}$ the *m-th SGC extension* of A. At first glance, one would expect that the algebras $A^{[m]}$ become complicated very quickly as m increases. Nevertheless, Yamagata observed that in the case when A is hereditary, every $A^{[m]}$ has finite global dimension and large dominant dimension. Although he did not calculate their explicit values, it is implicit from his claim that the difference between the global dimension between consecutive terms are at most 2, and so is the dominant dimension. This motivates to actually try to determine the explicit difference, which could lead to finding higher Auslander algebras in the $A^{[m]}$'s.

It turns out that in most cases, $A^{[m]}$ is Morita equivalent to another algebra $A^{(m)}$, where calculation of these homological dimensions can be done easily. The algebra $A^{(m)}$, which is first studied by [1], is called the *m-th replication* of A and is defined as the matrix algebra

$$\begin{pmatrix} A & 0 & 0 & \cdots & 0 \\ DA & A & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & DA & A \end{pmatrix}.$$

More precisely, we have the following result.

Theorem 1. *Let A be a finite dimensional algebra over a field k.*

(1) *If $\mathrm{Hom}_A(DA, A) = 0$, where DA is the k-linear dual of A, then the m-th SGC extension $A^{[m]}$ is Morita equivalent to the m-th replicated algebra $A^{(m)}$.*

(2) *The following conditions are equivalent.*
- (i) *A is representation-finite.*
- (ii) *$A^{(m)}$ is a higher Auslander algebra for some positive integer m.*
- (ii') *$A^{[m]}$ is a higher Auslander algebra for some positive integer m.*
- (iii) *$A^{(m)}$ is a higher Auslander algebra for infinitely many positive integers m.*
- (iii') *$A^{[m]}$ is a higher Auslander algebra for infinitely many positive integers m.*

Let us briefly explain the strategy to show (1), and some parts of (2). For details, we refer the reader to [2].

(1) For $m = 1$, $A^{[m]} := \mathrm{End}_A(A \oplus DA)$ can be written as

$$\begin{pmatrix} \mathrm{Hom}_A(A, A) & \mathrm{Hom}_A(DA, A) \\ \mathrm{Hom}_A(A, DA) & \mathrm{Hom}_A(DA, DA) \end{pmatrix} \cong \begin{pmatrix} A & 0 \\ DA & A \end{pmatrix} \cong A^{(1)}.$$

Then we proceed by induction on m.

Consider the row matrix $I_0 := (DA, 0, \ldots, 0)$ with $(m+1)$ entries, which is a natural (right) injective $A^{(m)}$-module. Hence, we get that $\mathrm{Hom}_{A^{(m)}}(A^{(m)}, I_0) = \mathrm{Hom}_A(A, DA) = DA$, $\mathrm{Hom}_{A^{(m)}}(I_0, A^{(m)}) = \mathrm{Hom}_A(DA, A) = 0$ and $\mathrm{Hom}_{A^{(m)}}(I_0, I_0) = \mathrm{Hom}_A(DA, DA) = A$. Putting these information together we get that

$$\mathrm{End}_{A^{(m)}}(A^{(m)} \oplus I_0) \cong \begin{pmatrix} A^{(m)} & DA \\ 0 & A \end{pmatrix} \cong A^{(m+1)}.$$

Observe that the (right) projective-injective direct summand of $A^{(m)}$ is given by the first m-th rows of the defining matrix. This means that the additive closure $\mathrm{add}(A^{(m)} \oplus DA^{(m)})$ coincide with the additive closure $\mathrm{add}(A^{(m)} \oplus I_0)$. By the induction hypothesis, $A^{[m+1]}$ (the SGC-extension of $A^{[m]}$) is Morita equivalent to $\mathrm{End}_{A^{(m)}}(A^{(m)} \oplus DA^{(m)})$; hence, it is also equivalent to $\mathrm{End}_{A^{(m)}}(A^{(m)} \oplus I_0)$. The above calculation then completes the proof of (1).

(2) When A is uniserial (hence, representation-finite), one can calculate the global and dominant dimensions of $A^{[m]}$ inductively and see that (ii') and (iii') hold. Otherwise, it follows from (1) that $A^{(m)}$ and $A^{[m]}$ coincide up to Morita equivalence, so we only need to calculate the global and dominant dimensions of $A^{(m)}$ to complete the proof.

To do this, we write down explicitly the minimal injective coresolution of (the projective non-injective indecomposable direct summands of) $A^{(m)}$. For the details on how this is done, and the formulae of these homological dimensions, we refer the reader to [2].

2. Generalising the result on replicated algebras

Recall that the Serre functor ν on a triangulated category $\mathcal{D}$ is an auto-equivalence so that $\mathrm{Hom}_{\mathcal{D}}(X, Y) \cong D\mathrm{Hom}_{\mathcal{D}}(Y, \nu X)$. In the case when A is Iwanaga-Gorenstein (see definition below), we have a Serre functor $\nu = DA \otimes_A^L -$ on the category $\mathcal{D} = \mathrm{perf}(A)$ of perfect complexes.

The essential ingredient needed to write down the minimal injective coresolution of $A^{(m)}$ is that for all $i \in \mathbb{Z}$ and indecomposable projective A-module P, $\nu^i(P)$ is a stalk complex. Once we have noticed this, it is not difficult to generalise the equivalence of (i), (ii), (iii) in the previous theorem to a more general class of algebras.

Definition 2. Let A be a finite dimensional algebra over a field k.

1. A is *Iwanaga-Gorenstein* if the injective dimension of the left and right regular modules are both finite.
2. A is *minimal Auslander-Gorenstein* if it is Iwanaga-Gorenstein with the injective dimension of the (left or right) regular module coinciding with its dominant dimension.
3. We say that A is *Serre-formal* if it is Iwanaga-Gorenstein and $\nu^k(A)$ is isomorphic to a direct sum of stalk complexes of A-module in the derived category of $\mathrm{mod}A$, for all integers k.
4. We say that A is *periodically Serre-formal* if it is Serre-formal, and there is some integers k so that $\nu^k(P)$ is isomorphic to a stalk complex whose underlying module is projective.

Iwanaga-Gorenstein algebras are somewhat well-known to be the class of algebras whose representation theory exhibit both features of self-injective algebras (via the so-called Gorenstein projective modules) and finite global dimensions algebras. The other three notions can be thought as generalisations of higher Auslander algebras, hereditary algebras, and representation-finite hereditary algebras, respectively.

Example 3.
1. Algebras with finite global dimension and self-injective algebras are Iwanaga-Gorenstein.
2. Higher Auslander algebras are minimal Auslander-Gorenstein.

(3) Hereditary algebras and self-injective algebras are Serre-formal. Canonical algebras of Geigle-Lenzing are also Serre-formal.
(4) Representation-finite algebras and self-injective algebras are periodically Serre-formal.

Now we can state the generalisation of the previous theorem.

Theorem 4. *Let A be a finite dimensional algebra over a field k.*

- $A^{(m)}$ *is Iwanaga-Gorenstein.*
- *The following are equivalent.*
  (i) A *is periodically Serre-formal.*
  (ii) $A^{(m)}$ *is a minimal Auslander-Gorenstein algebra for some positive integer* m.
  (iii) $A^{(m)}$ *is a minimal Auslander-Gorenstein algebra for infinitely many positive integers* m.

Finally, we remark that the precise dominant and injective dimensions of regular modules can be computed if one can record the degree of the stalk complexes appearing in $\nu^k(A)$ for all $k \geq 0$. In certain cases, such as the case of hereditary algebras, these degrees are constant over k for representation-infinite ones, and "almost constant" for representation-finite case.

Also, when A is periodically Serre-formal, one can determine explicitly for which m the replicated algebra $A^{(m)}$ is minimal Auslander-Gorenstein. It turns out that such an m, as well as the coinciding homological dimensions of $A^{(m)}$, can be expressed in terms of the so-called fractional Calabi-Yau dimension of A, which exists for a Serre-formal algebra precisely when it is periodically Serre-formal. For details, we refer the interested audience to [2].

References

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