RELATIVE NON-COMMUTNG GRAPH OF A FINITE RING

JUTIREKHA DUTTA AND DHIREN K. BASNET*

Abstract. Let S be a subring of a finite ring R and $C_R(S) = \{r \in R : rs = sr \ \forall \ s \in S\}$. The relative non-commuting graph of the subring S in R, denoted by $\Gamma_{S,R}$, is a simple undirected graph whose vertex set is $R \setminus C_R(S)$ and two distinct vertices a, b are adjacent if and only if a or $b \in S$ and $ab \neq ba$. In this paper, we discuss some properties of $\Gamma_{S,R}$, determine diameter, girth, some dominating sets and chromatic index for $\Gamma_{S,R}$. Also, we derive some connections between $\Gamma_{S,R}$ and the relative commuting probability of S in R. Finally, we show that the relative non-commuting graphs of two relative $\mathbb{Z}$-isoclinic pairs of rings are isomorphic under some conditions.

Key Words: Non-commuting graph, Commuting probability, $\mathbb{Z}$-isoclinism.

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1. Introduction

Let R be a finite ring with subring S. Let $C_R(S) = \{r \in R : rs = sr \ \forall \ s \in S\}$. The relative non-commuting graph of the subring S in R, denoted by $\Gamma_{S,R}$, is defined as a simple undirected graph whose vertex set is $R \setminus C_R(S)$ and two distinct vertices a, b are adjacent if and only if a or $b \in S$ and $ab \neq ba$. For $S = R$, we have $\Gamma_{S,R} = \Gamma_R$, the non-commuting graph of R. The notion of non-commuting graph of a finite ring was introduced by Erfanian et al. [8] in the year 2015. The study of algebraic structures by means of graph theoretical properties became more popular during the last decade (see [1, 2, 3, 4, 11] etc.). Motivated by the works of Erfanian et al. [12], in this paper, we obtain some graphs that are not isomorphic to $\Gamma_{S,R}$ for any ring R with subring S. We also determine diameter, girth, some dominating sets and chromatic index for $\Gamma_{S,R}$ and derive some connections between $\Gamma_{S,R}$ and the relative commuting probability of S in R. Recall that the relative commuting probability of a subring S in a finite ring R, denoted by $\Pr(S, R)$, is the probability that a randomly chosen pair of elements, one from S and the other from R commute. That is

$$\Pr(S, R) = \frac{|\{(s, r) \in S \times R : sr = rs\}|}{|S||R|}.$$

This notion was introduced and studied in [7]. Note that $\Pr(R, R)$ is the commuting probability of R, a notion introduced by MacHale [10]. In the last section, we show that the relative non-commuting graphs of two relative $\mathbb{Z}$-isoclinic pairs of rings are isomorphic under some conditions.

For a graph $\mathcal{G}$, we write $V(\mathcal{G})$ and $E(\mathcal{G})$ to denote the set of vertices and the set of edges of $\mathcal{G}$ respectively. We write $\deg(v)$ to denote the degree of a vertex v, which is the

*Corresponding author.

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number of edges incident on v. Let $\mathrm{diam}(\mathcal{G})$ and $\mathrm{girth}(\mathcal{G})$ be the diameter and girth of a graph $\mathcal{G}$ respectively. Recall that $\mathrm{diam}(\mathcal{G}) = \max\{d(x, y) : x, y \in V(\mathcal{G})\}$, where $d(x, y)$ is the length of the shortest path from x to y; and $\mathrm{girth}(\mathcal{G})$ is the length of the shortest cycle obtained in $\mathcal{G}$. A graph $\mathcal{G}$ is called connected if there is a path between every pair of vertices. A star graph is a tree on n vertices in which one vertex has degree $n-1$ and the others have degree 1. A bipartite graph is a graph whose vertex set can be partitioned into two disjoint parts in such a way that the two end vertices of every edge lie in different parts. A complete bipartite graph is a bipartite graph such that two vertices are adjacent if and only if they lie in different parts. A complete graph is a graph in which every pair of distinct vertices is adjacent. Throughout the paper R denotes a finite non-commutative ring.

2. Some properties of $\Gamma_{S,R}$

Let S be a subring of a ring R, $r \in R$ and $A \subseteq R$. We write $C_R(r) := \{x \in R : xr = rx\}$, $C_S(r) := C_R(r) \cap S$ and $C_R(A) = \{x \in R : xa = ax \, \forall \, a \in A\}$. Note that $C_R(r)$ and $C_S(r)$ are subrings of R. Also $\underset{r \in R}{\cap} C_R(r) := Z(R)$ is the center of R. We begin this section with the following useful result.

Proposition 1. *Let S be a non-commutative subring of a ring R. Then*

(1) $\deg(r) = |R| - |C_R(r)|$ *if* $r \in V(\Gamma_{S,R}) \cap S$.
(2) $\deg(r) = |S| - |C_S(r)|$ *if* $r \in V(\Gamma_{S,R}) \cap (R \setminus S)$.
(3) $\Gamma_{S,R}$ *is connected.*

Proof. The proofs of part (a) and (b) follow from the definition of $\Gamma_{S,R}$. For part (c), suppose $\Gamma_{S,R}$ has an isolated vertex, namely v. Then $\deg(v) = |R| - |C_R(v)| = 0$ or $|S| - |C_S(v)| = 0$ for $v \in S$ or $v \in R \setminus S$. Thus, in both cases $v \in C_R(S)$, a contradiction. □

In the following theorems we shall show that if $\mathcal{G}$ is a star graph or a complete bipartite graph then $\mathcal{G}$ can not be realized by $\Gamma_{S,R}$ for any subring S of a ring R. Also, $\Gamma_{S,R}$ is not an n-regular graph for any proper subring S of a ring R, where n is a square free odd positive integer.

Theorem 2. *Let S be a non-commutative subring of a ring R. Then $\Gamma_{S,R}$ is not a star graph.*

Proof. Suppose, $\Gamma_{S,R}$ is a star graph, where S is a non-commutative subring of R. Then all but one vertices of $\Gamma_{S,R}$ have degree 1. Let v be a vertex of $\Gamma_{S,R}$ having degree 1. Then, by Proposition 1, we have $[R : C_R(v)] = |R|/(|R|-1)$ or $[S : C_S(v)] = |S|/(|S|-1)$ according as $v \in S$ or $v \in R \setminus S$; which is absurd. Hence the result follows. □

Theorem 3. *Let S be a proper non-commutative subring of a ring R. Then $\Gamma_{S,R}$ is not complete bipartite.*

Proof. Let $\Gamma_{S,R}$ be a complete bipartite graph. Then, there exist two disjoint subsets S_1 and S_2 of $V(\Gamma_{S,R})$ such that $|S_1| + |S_2| = |R| - |C_R(S)|$. Therefore, $S \cap S_1 = \phi$ or $S \cap S_2 = \phi$. So, $S \subseteq S_2$ or $S \subseteq S_1$. Without loss of generality we may assume that $S \subseteq S_1$. Then, for $v \in S_1$ we have $vs = sv$ for all $s \in S \setminus C_R(S)$. Thus, $v \in C_R(S)$, a contradiction. Hence, the theorem follows. □

Theorem 4. *Let S be a proper non-commutative subring of a ring R. Then $\Gamma_{S,R}$ is not an n-regular graph for any square free odd positive integer n.*

Proof. Let $\Gamma_{S,R}$ be an n-regular graph. Suppose, $n = p_1p_2 \dots p_m$, where p_i's are distinct odd primes. If $v \in V(\Gamma_{S,R}) \cap S$ then, by Proposition 1, we have

$$n = \deg(v) = |R| - |C_R(v)| = |C_R(v)|([R : C_R(v)] - 1).$$

Here, $|C_R(v)| \neq 1$, as $0, v \in C_R(v)$. Thus $|C_R(v)| = \prod\limits_{p_i \in Q} p_i$ and $[R : C_R(r)] - 1 = \prod\limits_{p_j \in P\setminus Q} p_j$, where $Q \subseteq \{p_1, p_2, \dots, p_m\} = P$. So, $|R| = \prod\limits_{p_i \in Q} p_i(\prod\limits_{p_j \in P\setminus Q} p_j + 1)$. If $r \in R \setminus S$ then, using similar argument, we have $|S| = \prod\limits_{p_i \in T} p_i(\prod\limits_{p_j \in P\setminus T} p_j + 1)$, where $T \subseteq P$. So, $\prod\limits_{p_i \in T-(T\cap Q)} p_i(\prod\limits_{p_j \in P\setminus T} p_j + 1)$ divides $\prod\limits_{p_j \in P\setminus Q} p_j + 1$, which is not possible. Hence, the theorem follows. □

We conclude this section showing that a complete graph can not be realized by $\Gamma_{S,R}$ for a subring S of a ring R with unity.

Theorem 5. *Let R be a ring with unity and S a subring of R. Then $\Gamma_{S,R}$ is not complete.*

Proof. Suppose that there exists a subring S of R with unity such that $\Gamma_{S,R}$ is complete. Then, for any $s \in V(\Gamma_{S,R}) \cap S$ we have

$$\deg(s) = |V(\Gamma_{S,R})| - 1 = |R| - |C_R(S)| - 1.$$

By Proposition 1, we have $|R| - |C_R(s)| = |R| - |C_R(S)| - 1$. This gives $|C_R(S)| = 1$ and $|C_R(s)| = 2$, which is not possible, since R is a ring with unity. Hence, the result follows. □

3. Diameter, girth, dominating set and chromatic index

In this section, we obtain diameter, girth, some dominating sets and chromatic index of the graph $\Gamma_{S,R}$.

Theorem 6. *Let S be a non-commutative subring of a ring R. If $Z(S) = \{0\}$ then $\operatorname{diam}(\Gamma_{S,R}) = 2$ and $\operatorname{girth}(\Gamma_{S,R}) = 3$.*

Proof. Suppose, v_1 and v_2 are two vertices of $\Gamma_{S,R}$ such that they are not adjacent. So, there exist vertices $s_1, s_2 \in S$ such that $v_1s_1 \neq s_1v_1$ and $v_2s_2 \neq s_2v_2$. If v_2 is adjacent to s_1 or v_1 is adjacent to s_2, then $d(v_1, v_2) = 2$. Suppose that both are not adjacent, that is $v_1s_2 = s_2v_1$ and $v_2s_1 = s_1v_2$. Then $s_1 + s_2$ is adjacent to v_1 and v_2, which give $d(v_1, v_2) = 2$. Therefore, $\operatorname{diam}(\Gamma_{S,R}) = 2$.

In order to determine $\operatorname{girth}(\Gamma_{S,R})$, suppose that $v, s \in V(\Gamma_{S,R})$ where $s \in S$ and v, s are adjacent. So, there exist $v_1, v_2 \in V(\Gamma_{S,R})$ such that v and s are adjacent to v_1 and v_2 respectively. If v, v_2 or s, v_1 are adjacent then $\{v, s, v_2\}$ or $\{v, s, v_1\}$ is a cycle of length 3 in $\Gamma_{S,R}$. If both are not adjacent then $v_1 + v_2$ is adjacent to v and s. Therefore, $\{v, s, v_1 + v_2\}$ is a cycle of length 3 in $\Gamma_{S,R}$. Hence, $\operatorname{girth}(\Gamma_{S,R}) = 3$. □

Let $\mathcal{G}$ be a graph and D a subset of $V(\mathcal{G})$ such that every vertex not in D is adjacent to at least one member of D then D is called the dominating set for $\mathcal{G}$. It is obvious that $V(\mathcal{G})$ is a dominating set for $\mathcal{G}$. Again, it is easy to see that for any non-commutative subring S of R, the set $S \setminus Z(S)$ is a dominating set for $\Gamma_{S,R}$. Let A and B be two subsets of R. We define $A + B := \{a + b : a \in A, b \in B\}$. Then it can be seen that $(S + C_R(S)) \setminus C_R(S)$ is a dominating set for $\Gamma_{S,R}$ if S is a non-commutative subring of a finite ring R. The following propositions also give dominating sets for $\Gamma_{S,R}$.

Proposition 7. *Let S be a subring of a ring R and $A \subseteq V(\Gamma_{S,R})$. Then A is a dominating set for $\Gamma_{S,R}$ if and only if $C_R(A) \subseteq A \cup C_R(S)$.*

Proof. Suppose, A is a dominating set for $\Gamma_{S,R}$ and $v \in V(\Gamma_{S,R})$ such that $v \in C_R(A)$. If $v \notin A$ then there exists an element $a \in A$ such that $va \neq av$, a contradiction.

Conversely, we suppose that $C_R(A) \subseteq A \cup C_R(S)$. Let $v \in V(\Gamma_{S,R})$ such that $v \notin A$. Suppose that $va = av$ for all $a \in A$. Then $v \in C_R(A)$ and so $v \in A \cup C_R(S)$. Thus, $v \in A$, a contradiction. Hence, A is a dominating set for $\Gamma_{S,R}$. □

Proposition 8. *Let R be a ring with unity and S a subring of R. If $L = \{s_1, s_2, \ldots, s_n\}$ is a generating set for S and $L \cap C_R(S) = \{s_{m+1}, \ldots, s_n\}$ then $K = \{s_1, s_2, \ldots, s_m\} \cup \{s_1 + s_{m+1}, s_1 + s_{m+2}, \ldots, s_1 + s_n\}$ is a dominating set for $\Gamma_{S,R}$.*

Proof. Clearly, $K \subseteq V(\Gamma_{S,R})$. Let $v \in V(\Gamma_{S,R})$ such that $v \notin L$. If $v \in S$ then there exists an element $s = \beta_i {s_1}^{\alpha_{1i}} {s_2}^{\alpha_{2i}} \ldots {s_d}^{\alpha_{di}}$, where $\beta_i \in \mathbb{Z}$, $\alpha_{ji} \in \mathbb{N} \cup \{0\}$ and $s_j \in L$ such that $vs \neq sv$. Therefore, $vs_i \neq s_i v$ for some $1 \leq i \leq m$ and so, v is adjacent to s_i.

If $v \in R \setminus S$ then there exists an element $u = \gamma_i {s_1}^{\alpha_{1i}} {s_2}^{\alpha_{2i}} \ldots {s_p}^{\alpha_{pi}}$, where $\gamma_i \in \mathbb{Z}$, $\alpha_{li} \in \mathbb{N} \cup \{0\}$ and $s_l \in L$ such that $vu \neq uv$. If $vs_i \neq s_i v$ for some $1 \leq i \leq m$ then v is adjacent to s_i. Otherwise, $vs_i = s_i v$ for all $1 \leq i \leq m$. So, there exists an element s_l for some $m + 1 \leq l \leq n$ such that $vs_l \neq s_l v$. Therefore, v is adjacent to $s_1 + s_l$. Hence, the proposition. □

An edge coloring of a graph $\mathcal{G}$ is an assignment of "colors" to the edges of the graph so that no two adjacent edges have the same color. The chromatic index of a graph denoted by $\chi'(\mathcal{G})$ and is defined as the minimum number of colours needed for a colouring of $\mathcal{G}$. Let $\triangle$ be the maximum vertex degree of $\mathcal{G}$, then Vizing's theorem [6] gives $\chi'(\mathcal{G}) = \triangle$ or $\triangle + 1$. Thus, Vizing's theorem divides the graphs into two classes according to their chromatic index. Graphs satisfying $\chi'(\mathcal{G}) = \triangle$ are called graphs of class 1 and those with $\chi'(\mathcal{G}) = \triangle + 1$ are called graphs of class 2. Following theorem shows that $\Gamma_{R,R}$ is of class 2.

Theorem 9. *Let R be a ring. Then the non-commuting graph $\Gamma_{R,R}$ is of class 2.*

Proof. Clearly, $\triangle \leq |R| - |Z(R)| - 1$. If $\chi'(\Gamma_{S,R}) = \triangle$ then $\chi'(\Gamma_{S,R}) \leq |R| - |Z(R)| - 1$, which is not true for the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in Z_2 \right\}$. Hence, $\Gamma_{R,R}$ is class of 2. □

We conclude this section with the following conjecture.

Conjecture 10. *Let S be a proper non-commutative subring of R. Then the relative non-commuting graph $\Gamma_{S,R}$ is of class 1.*

4. Relative non-commuting graphs and $\Pr(S, R)$

In this section, we give some connections between $\Gamma_{S,R}$ and $\Pr(S, R)$, where S is a subring of a finite ring R. We start with the following result.

Theorem 11. *Let S be a subring of a ring R. Then the number of edges of $\Gamma_{S,R}$ is*

$$|E(\Gamma_{S,R})| = |S||R|(1 - \Pr(S, R)) - \tfrac{|S|^2}{2}(1 - \Pr(S)).$$

Proof. Let $I = \{(r_1, r_2) \in S \times R : r_1r_2 \neq r_2r_1\}$ and $J = \{(r_1, r_2) \in R \times S : r_1r_2 \neq r_2r_1\}$. Therefore, we have $|I| = |S||R| - |\{(r_1, r_2) \in S \times R : r_1r_2 = r_2r_1\}| = |S||R| - |S||R| \Pr(S, R) = |J|$ and so $|I \cap J| = |\{(a, b) \in S \times S : ab \neq ba\}| = |S|^2 - |S|^2 \Pr(S)$. Thus, the result follows from the fact that $|E(\Gamma_{S,R})| = \frac{1}{2}|I \cup J|$. $\square$

The above theorem shows that lower or upper bounds for $\Pr(S)$ and $\Pr(S, R)$ will give lower or upper bounds for $|E(\Gamma_{S,R})|$ and vice-versa. More bounds for $|E(\Gamma_{S,R})|$ are obtained in the next few results.

Proposition 12. *Let S be a subring of a ring R. Then*

$$|E(\Gamma_{S,R})| \geq \frac{1}{2}|S||R| - \frac{1}{4}|S|^2 - \frac{1}{4}|Z(S)||R| - \frac{1}{4}|S||C_R(S)| + \frac{1}{4}|Z(S)||S|.$$

Proof. Let $A = V(\Gamma_{S,R}) \cap S$ and $B = V(\Gamma_{S,R}) \cap (R \setminus S)$. Therefore, $|A| = |S| - |Z(S)|$ and $|B| = |R| - |S| - |C_R(S)| + |Z(S)|$. So, we have

$$\begin{aligned} 2|E(\Gamma_{S,R})| &= \sum_{v \in V(\Gamma_{S,R})} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in B} \deg(v) \\ &= \sum_{v \in A} (|R| - |C_R(r)|) + \sum_{v \in B} (|S| - |C_S(r)|) \\ &\geq |A||R| - \frac{|A||R|}{2} - |B||S| - \frac{|S||B|}{2}. \end{aligned}$$

Thus, putting the values of $|A|$ and $|B|$, we get the required result. $\square$

We conclude this section with some consequences of Theorem 11.

Proposition 13. *Let S be a non-commutative subring of a ring R and p the smallest prime dividing $|R|$. Then*

$$|E(\Gamma_{S,R})| \leq |S|(|R| - \frac{3|S|}{16} - p) - |Z(R) \cap S|(|R| - p)$$

Proof. By [7, Theorem 2.5], we have

$$\frac{|Z(R) \cap S|}{|S|} + \frac{p(|S| - |Z(R) \cap S|)}{|S||R|} \leq \Pr(S, R). \tag{4.1}$$

Now, using (4.1) and the fact that $\Pr(S) \leq \frac{5}{8}$ in Theorem 11 we get the required result. $\square$

Proposition 14. *Let S be a non-commutative subring of a ring R. Then*

$$|E(\Gamma_{S,R})| \geq -\frac{3|S|^2}{16} + \frac{3|S||R|}{8}.$$

Proof. Using [7, Theorem 2.2], we have that $\Pr(S,R) \leq \Pr(S) \leq \frac{5}{8}$. Therefore, $1 - \Pr(S,R) \geq 1 - \Pr(S) \geq \frac{3}{8}$. Hence, putting these results in Theorem 11, we get the required proposition. $\square$

Proposition 15. *Let S be a non-commutative subring of a ring R. If $|C_R(S)| = 1$ then*
$$2|R|\Pr(S,R) - |S|\Pr(S) \neq -2\frac{|R|}{|S|} + \frac{4}{|S|} + 2|R| - |S|.$$

Proof. Suppose there exists a finite ring R with non-commutative subring S such that $|C_R(S)| = 1$ and
$$2|R|\Pr(S,R) - |S|\Pr(S) = -2\frac{|R|}{|S|} + \frac{4}{|S|} + 2|R| - |S|.$$
Then the above equation, in view of Theorem 11, gives
$$|E(\Gamma_{S,R})| = |R| - |C_R(S)| - 1 = |V(\Gamma_{S,R})| - 1.$$
This shows that there is a finite non-commutative ring R with non commutative subring S such that $\Gamma_{S,R}$ is a star graph, which is not possible (by Theorem 2). Hence, the proposition follows. $\square$

5. Relative non-commuting graph and relative $\mathbb{Z}$-isoclinism

In 1940, Hall [9] introduced the notion of isoclinism between two groups. Following Hall, Buckley et al. [5] introduced the concept of $\mathbb{Z}$-isoclinism between two rings. Recently, Dutta et al. [7] introduced the concept of relative $\mathbb{Z}$-isoclinism between two pairs of rings. For a subring S of R, $[S,R]$ is the subgroup of $(R,+)$ generated by all commutators $[s,r], s \in S, r \in R$. Let S_1 and S_2 be two subrings of the rings R_1 and R_2 respectively. Recall that a pair of rings (S_1,R_1) is said to be relative $\mathbb{Z}$-isoclinic to a pair of rings (S_2,R_2) if there exist additive group isomorphisms $\phi : \frac{R_1}{Z(R_1)\cap S_1} \to \frac{R_2}{Z(R_2)\cap S_2}$ such that $\phi(\frac{S_1}{Z(R_1)\cap S_1}) = \frac{S_2}{Z(R_2)\cap S_2}$ and $\psi : [S_1,R_1] \to [S_2,R_2]$ such that $\psi([s_1,r_1]) = [s_2,r_2]$ whenever $\phi(s_1 + (Z(R_1)\cap S_1)) = s_2 + (Z(R_2)\cap S_2)$ and $\phi(r_1 + (Z(R_1)\cap S_1)) = r_2 + (Z(R_2)\cap S_2)$ where $s_1 \in S_1, s_2 \in S_2, r_1 \in R_1, r_2 \in R_2$. Such pair of mappings (ϕ,ψ) is called a relative $\mathbb{Z}$-isoclinism from (S_1,R_1) to (S_2,R_2). In this section, we have the following main result.

Theorem 16. *Let S_1 and S_2 be two subrings of the finite rings R_1 and R_2 respectively. Let the pairs (S_1,R_1) and (S_2,R_2) are relative $\mathbb{Z}$-isoclinic. Then $\Gamma_{S_1,R_1} \cong \Gamma_{S_2,R_2}$ if $|Z(R_1)\cap S_1| = |Z(R_2)\cap S_2|$ and $|Z(R_1)| = |Z(R_2)|$.*

Proof. Suppose (ϕ,ψ) is a relative $\mathbb{Z}$-isoclinism between (S_1,R_1) and (S_2,R_2). If $|Z(R_1)\cap S_1| = |Z(R_2)\cap S_2|$ and $|Z(R_1)| = |Z(R_2)|$ then $|S_1| = |S_2|, |\frac{R_1}{Z(R_1)}| = |\frac{R_2}{Z(R_2)}|, |Z(R_1)\setminus S_1| = |Z(R_2)\setminus S_2|$ and $|S_1 \setminus Z(R_1)| = |S_2 \setminus Z(R_2)|$. Now, by second isomorphism theorem (of groups), we have $\frac{S_1}{S_1\cap Z(R_1)} \cong \frac{S_1+Z(R_1)}{Z(R_1)}$. Let $\{s_1,s_2,\dots,s_m\}$ be a transversal for $\frac{S_1+Z(R_1)}{Z(R_1)}$. So, the set $\{s_1,s_2,\dots,s_m\}$ can be extended to a transversal for $\frac{R_1}{Z(R_1)}$. Suppose, $\{s_1,s_2,\dots,s_m,r_{m+1},\dots,r_n\}$ is a transversal for $\frac{R_1}{Z(R_1)}$. Similarly, we can find a transversal $\{s'_1,s'_2,\dots,s'_m,$
$r'_{m+1},\dots,r'_n\}$ for $\frac{R_2}{Z(R_2)}$ such that $\{s'_1,s'_2,\dots,s'_m\}$ is a transversal for $\frac{S_2+Z(R_2)}{Z(R_2)} \cong \frac{S_2}{S_2\cap Z(R_2)}$.

Let ϕ be defined as $\phi(s_i + Z(R_1)) = s'_i + Z(R_2)$, $\phi(r_j + Z(R_1)) = r'_j + Z(R_2)$ for $1 \leq i \leq m, m+1 \leq j \leq n$ and let the one-to-one correspondence $\theta : Z(R_1) \to Z(R_2)$ maps elements of S_1 to S_2. Therefore, $|C_{R_1}(S_1)| = |C_{R_2}(S_2)|$. Let us define a map $\alpha : R_1 \to R_2$ such that $\alpha(s_i + z) = s'_i + \theta(z)$, $\alpha(r_j + z) = r'_j + \theta(z)$ for $1 \leq i \leq m, m+1 \leq j \leq n$ and $z \in Z(R_1)$. Then α is a bijection. This gives that α is also a bijection from $R_1 \setminus C_{R_1}(S_1)$ to $R_2 \setminus C_{R_2}(S_2)$. Suppose u, v are adjacent in Γ_{S_1,R_1}. Then $u \in S_1$ or $v \in S_1$, say $u \in S_1$. So, $[u, v] \neq 0$, therefore $[s_i + z, r + z_1] \neq 0$, where $u = s_i + z, v = r + z_1$ for some $z, z_1 \in Z(R_1)$, $r \in \{s_1, s_2, \ldots, s_m, r_{m+1}, \ldots, r_n\}$ and $1 \leq i \leq m$. Thus $[s'_i + \theta(z), r + \theta(z_1)] \neq 0$, where $\theta(z), \theta(z_1) \in Z(R_2)$ and so, $\alpha(u)$ and $\alpha(v)$ are adjacent. Hence, the theorem. □

We conclude the paper with the following consequence of Theorem 16.

Corollary 17. *Let R be a ring with subrings S and T such that (S, R) is relative $\mathbb{Z}$-isoclinic to (T, R). Then $\Gamma_S \cong \Gamma_T$ if $|Z(R) \cap S| = |Z(R) \cap T|$.*

References

[1] A. Abdollahi, *Engel graph associated with a group*, J. Algebra **318** (2007), 680–691.

[2] A. Abdollahi, *Commuting graph of full matrix rings over finite fields*, Linear Algebra Appl. **428** (2008), 2947–2954.

[3] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra **298** (2006), 468–492.

[4] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

[5] S. M. Buckley, D. Machale and A. Ní Shé, *Finite rings with many commuting pairs of elements*, Preprint.

[6] R. Diestel, *Graph Theory*. Springer-Verlag, New York 1997, electronic edition 2000.

[7] J. Dutta , D. K. Basnet and R. K. Nath, *On commuting probability of finite rings*, Indag. Math. (N.S.) **28(2)**(2017), 372–382.

[8] A. Erfanian, K. Khashyarmanesh and Kh. Nafar, *Non-commuting graphs of rings*, Discrete Math. Algorithms Appl. **7(3)** (2015), 1550027-1–1550027-7.

[9] P. Hall, *The classification of prime power groups*, J. Reine Angew. Math. **182** (1940), 130–141.

[10] D. MacHale, *Commutativity in finite rings*, Amer. Math. Monthly **83** (1976), 30–32.

[11] G. R. Omidi and E. Vatandoost *On the commuting graph of rings*, J. Algebra Appl. **10(3)** (2011), 521–527.

[12] B. Tolue and A. Erfanian, *Relative non-commuting graph of a finite group*, J. Algebra Appl. **12(2)** (2013), 1250157-1–1250157-11.

Department of Mathematical Sciences
Tezpur University
Napaam-784028, Sonitpur, Assam, India
E-mail address: jutirekhadutta@yahoo.com and dbasnet@tezu.ernet.in