CLASSIFICATIONS OF EXACT STRUCTURES AND COHEN-MACAULAY-FINITE ALGEBRAS

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Abstract. We give a classification of all exact structures on a given additive category. Using this, we investigate the structure of an exact category with finitely many indecomposables. As an application of this, we give an explicit classification of Cohen-Macaulay-finite Iwanaga-Gorenstein algebras.

1. Introduction

Exact categories, in the sense of Quillen, have been playing an important role in the representation theory of algebras. In general, an additive category has many exact structures. Recently, Rump [9] showed that every additive category has the largest exact structures, but no general description of exact structures was known. We give an explicit description of all exact structures on a given additive category $\mathcal{E}$ by using particular modules over $\mathcal{E}$.

Apart from the purely theoretical interest, another motivation for classifying exact structures comes from the "relative representation theory" of algebras, such as the Cohen-Macaulay representation theory. First, let k be a field and consider the "absolute" representation theory of finite-dimensional k-algebras, which investigates the category $\mathsf{mod}\,\Lambda$ of finitely generated Λ-modules. An algebra is *representation-finite* if there exists only finitely many indecomposable objects in $\mathsf{mod}\,\Lambda$ up to isomorphism. As for such algebras, there are bijections between the following classes, called an *Auslander correspondence*.

(1) Representation-finite finite-dimensional k-algebras Λ.
(2) Abelian Hom-finite k-categories $\mathcal{E}$ with finitely many indecomposables.
(3) Finite-dimensional k-algebras Γ satisfying $\mathsf{gl.dim}\,\Gamma \leq 2 \leq \mathsf{dom.dim}\,\Gamma$.

The map from (1) to (2) is given by $\Lambda \mapsto \mathsf{mod}\,\Lambda$, and from (3) to (2) by $\Gamma \mapsto \mathsf{proj}\,\Gamma$. Actually, the bijection between (2) and (3) is induced from the bijection between:

(2)′ Hom-finite Krull-Schmidt k-categories $\mathcal{E}$ with finitely many indecomposables.
(3)′ Finite-dimensional k-algebras Γ.

We call the algebra Γ in corresponding to $\mathcal{E}$ under the above bijection an *Auslander algebra* for $\mathcal{E}$. Under the bijection between (2)′ and (3)′, it is natural to guess that categorical properties of $\mathcal{E}$ and the homological property of Γ are related. In this viewpoint, the Auslander correspondence can be interpreted as saying that the abelianness of $\mathcal{E}$ can be characterized by the purely homological behavior of its Auslander algebra.

Now let us move to the relative situation. For a given algebra Λ, in addition to the module category $\mathsf{mod}\,\Lambda$, there are several important subcategories of it which have been

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investigated, e.g. the category $\mathsf{CM}\,\Lambda$ of Cohen-Macaulay Λ-modules. Usually, these "relative representation categories" are extension-closed in $\mathsf{mod}\,\Lambda$, thus naturally has the structure of an exact category. The main motivating question in our study is whether there exists a bijection between the following:

(1) *Relative* representation-finite algebras Λ.
(2) Hom-finite Krull-Schmidt k-categories $\mathcal{E}$ with finitely many indecomposables *which satisfy some categorical properties.*
(3) Finite-dimensional k-algebras *satisfying some homological properties.*

Here we fix some relative theory, and an algebra is relative representation-finite if its relative representation category has finitely many indecomposables. The maps are defined (if possible) as follows: For Λ in (1), we define $\mathcal{E}$ in (2) to be the relative representation category of Λ, and the map between (2) and (3) is a restriction of $(2)'$ and $(3)'$ above. Actually several bijections of this type are known, e.g. for one-dimensional orders [1], 1-cotilting modules and 1-Iwanaga-Gorenstein algebras [7].

In general, however, it often happens that the map from (1) to (2) is not injective. For example, there exist non-Morita-equivalent Iwanaga-Gorenstein algebras Λ and Λ' such that $\mathsf{CM}\,\Lambda$ and $\mathsf{CM}\,\Lambda'$ are equivalent. The problem is that the algebra Λ cannot be recovered from the additive structure of the representation category like $\mathsf{CM}\,\Lambda$. Nevertheless, we can usually recover Λ from the representation category of Λ *together with the exact structure on it.* Therefore, by using exact structures, we should modify and divide the question into the following.

(A) Construct a bijection between the "exact version" of (1) and (2), that is, characterize exact categories which are exact-equivalent to the representation category.
(B) Construct a bijection between the "exact version " of $(2)'$ and $(3)'$, that is, classify the exact structure on a given category in terms of its Auslander algebra.

Problem (A) is something like an Morita-type theory of exact categories, and was tackled with in [5]. In this article, we focus on Problem (B).

Remark 1. Although we concentrate on categories of finite type in this article, the completely similar argument works for general idempotent complete additive categories by using functor categorical method, except arguments about 2-regular simples. We refer the interested reader to [6] for details.

1.1. **Convention.** For simplicity, throughout this article, we fix a field k. All algebras are finite-dimensional k-algebras. All categories are skeletally small Hom-finite Krull-Schmidt k-categories, and all subcategories are assumed to be full, additive and closed under direct summands.

For an algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ (resp. $\mathsf{proj}\,\Lambda$) the category of finitely generated right Λ-modules (resp. finitely generated projective right Λ-modules).

For an object G in a category $\mathcal{E}$, we denote by $\mathsf{add}\,G$ the subcategory of $\mathcal{E}$ consisting of direct summands of finite direct sums of G. A category $\mathcal{E}$ is called *of finite type* if $\mathcal{E}$ has only finitely many indecomposable objects up to isomorphism, or equivalently, there exists $G \in \mathcal{E}$ such that $\mathcal{E} = \mathsf{add}\,G$. In this case such an object G is called an *additive generator* of $\mathcal{E}$.

2. Preliminaries on exact categories and Auslander algebras

We fix some terminology and recall basic properties of exact categories and auslander algebras. We refer the reader to [4] for the basics of exact categories.

Let $\mathcal{E}$ be an additive category. A complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ is a *kernel-cokernel pair* if f is a kernel of g and g is a cokernel of f. An *exact category* is a pair $(\mathcal{E}, F)$ consisting of an additive category $\mathcal{E}$ and a class F of kernel-cokernel pairs in $\mathcal{E}$ which satisfy some conditions (see [4]). In this case, we say that F is an *exact structure* for $\mathcal{E}$, and a complex in F is called a *conflation*.

Example 2. Let Λ be an algebra and $\mathcal{E}$ a subcategory of $\mathsf{mod}\,\Lambda$. If $\mathcal{E}$ is closed under extensions, then $\mathcal{E}$ has the natural exact structure, whose conflations are short exact sequences in $\mathsf{mod}\,\Lambda$ with all terms in $\mathcal{E}$.

As we mentioned in the introduction, categories of finite type are just finite-dimensional algebras, seen from another perspective:

Proposition 3. *There exists a bijection between the following classes.*

(1) *Equivalence classes of Hom-finite Krull-Schmidt categories $\mathcal{E}$ of finite type.*
(2) *Morita-equivalence classes of finite-dimensional k-algebras Γ.*

The map from (1) *to* (2) *is given by* $\Gamma := \operatorname{End}_{\mathcal{E}}(G)$*, where G is an additive generator of $\mathcal{E}$, and from* (2) *to* (1) *by* $\mathcal{E} := \mathsf{proj}\,\Gamma$.

In what follows, **Assumption** $(*)$ means that $\mathcal{E}$ is of finite type, G is an additive generator of $\mathcal{E}$ and $\Gamma := \operatorname{End}_{\mathcal{E}}(G)$. Our aim is to translate the information of exact structures on $\mathcal{E}$ into the homological behavior of Γ. To this purpose, the following equivalence called Auslander's *projectivization* plays an important role.

Lemma 4. *Assume* $(*)$*. Then we have the following equivalence and duality:*

(1) $P_{(-)} := \mathcal{E}(G, -) : \mathcal{E} \simeq \mathsf{proj}\,\Gamma$.
(2) $P^{(-)} := \mathcal{E}(-, G) : \mathcal{E} \simeq \mathsf{proj}\,\Gamma^{\mathrm{op}}$.

Moreover these satisfies $\operatorname{Hom}_\Gamma(P_{(-)}, \Gamma) \simeq P^{(-)}$.

Since an exact structure on $\mathcal{E}$ is a class of kernel-cokernel pairs in $\mathcal{E}$, let us investigate kernel-cokernel pairs in $\mathcal{E}$ in terms of Γ.

Proposition 5. *Assume* $(*)$*. Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a complex in $\mathcal{E}$ and put* $M := \operatorname{coker}(P_g) \in \mathsf{mod}\,\Gamma$.

(1) *f is a kernel of g if and only if the following is exact in* $\mathsf{mod}\,\Gamma$*. Thus* $\mathsf{pd}\, M_\Gamma \leq 2$.

$$0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M \to 0$$

(2) *Suppose* (1)*. Then (f, g) is a kernel-cokernel pair if and only if* $\operatorname{Ext}^{0,1}_\Gamma(M, \Gamma) = 0$.

Proof. We give a sketch of the proof. (1) is immediate from the definition of kernels in an additive category. By duality, g is a cokernel of f if and only if $0 \to P^Z \xrightarrow{P^g} P^Y \xrightarrow{P^f} P^X$ is exact. Since $P^{(-)} \simeq \operatorname{Hom}_\Gamma(P_{(-)}, \Gamma)$ and we are assuming (1), this is equivalent to $\operatorname{Ext}^{0,1}_\Gamma(M, \Gamma) = 0$. $\square$

As can be inferred from this, the following category will play a crucial role.

Definition 6. Let Γ be an algebra. A subcategory $\mathcal{C}_2(\Gamma)$ of $\mathsf{mod}\,\Gamma$ consists of modules M satisfying $\mathsf{pd}\,M_\Gamma \leq 2$ and $\mathrm{Ext}^{0,1}_\Gamma(M,\Gamma)=0$.

Under assumption $(*)$, each object in $\mathcal{C}_2(\Gamma)$ corresponds to a kernel-cokernel pair in $\mathcal{E}$. Therefore, it is natural to guess that exact structures on $\mathcal{E}$ correspond to a somewhat nice subcategories of $\mathcal{C}_2(\Gamma)$.

3. Main Results

Let Γ be an algebra. Recall that a subcategory $\mathcal{D}$ of $\mathsf{mod}\,\Gamma$ is called *Serre* if it is closed under extensions, submodules and quotients. Now we state the main result of this article.

Theorem 7. *Assume $(*)$. Then there exists a bijection between the following:*

(1) *Exact structures F on $\mathcal{E}$.*
(2) *Subcategories $\mathcal{D}$ of $\mathcal{C}_2(\Gamma)$ satisfying the following conditions.*
    (a) *$\mathcal{D}$ is Serre in $\mathsf{mod}\,\Gamma$.*
    (b) *$\mathrm{Ext}^2_\Gamma(\mathcal{D},\Gamma)$ is Serre in $\mathsf{mod}\,\Gamma^{\mathrm{op}}$.*

More precisely, suppose that F and $\mathcal{D}$ correspond to each other.

- *$M \in \mathsf{mod}\,\Gamma$ belongs to $\mathcal{D}$ if and only if there is a conflation $X \xrightarrow{f} Y \xrightarrow{g} Z$ in F such that $M \cong \operatorname{coker} P_g$.*
- *A complex $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ belongs to F if and only if $0 \to P_X \xrightarrow{P_f} P_Y \xrightarrow{P_g} P_Z \to M \to 0$ is exact for $M \in \mathcal{D}$.*

Since Serre subcategories of $\mathsf{mod}\,\Gamma$ are uniquely determined by sets of simple Γ-modules, we can describe exact structures more explicitly. Here the following notion is essential, which is similar to the behavior of the simple over 2-dimensional regular local ring.

Definition 8. Let Γ be an algebra. A simple Γ-module S is called 2-*regular* if it satisfies the following two conditions.

(1) $S \in \mathcal{C}_2(\Gamma)$, that is, $\mathsf{pd}\,S_\Gamma = 2$ and $\mathrm{Ext}^{0,1}_\Gamma(S,\Gamma)=0$.
(2) $\mathrm{Ext}^2_\Gamma(S,\Gamma)$ is a simple Γ^{op}-modules.

Assume $(*)$. As in the classical (functorial) Auslander-Reiten theory, 2-regular simple modules correspond to *almost split kernel-cokernel pairs* in $\mathcal{E}$. In order to help understand 2-regular simples visually, let us introduce the translation quiver $Q(\mathcal{E})$. The underlying quiver of $Q(\mathcal{E})$ is nothing but the usual quiver of $\mathcal{E}$, that is, vertices of $Q(\mathcal{E})$ are isomorphism classes of indecomposable objects in $\mathcal{E}$, and arrows are drawn depending on spaces of irreducible maps between objects in $\mathcal{E}$. Next we draw a *dotted arrow* $X \dashleftarrow Z$ if there exists an almost split kernel-cokernel pairs $X \to Y \to Z$ in $\mathcal{E}$. This happens precisely when the simple quotient S_Z of P_Z is a 2-regular simple Γ-module and $\mathrm{Ext}^2_\Gamma(S_Z,\Gamma)$ is the simple quotient of P^X.

By investigating simple modules corresponding to a Serre subcategory $\mathcal{D}$ of $\mathsf{mod}\,\Gamma$ in Theorem 7(2), we obtain the following description.

Theorem 9. *Assume $(*)$. Then we can add the following to Theorem 7.*

(3) *Sets $\mathbb{S}$ of 2-regular simple Γ-modules.*

(4) *Sets* $\mathbb{A}$ *of dotted arrows in* $Q(\mathcal{E})(= Q(\mathsf{proj}\,\Gamma))$.

Remark 10. Assume $(*)$ and suppose that $\mathcal{E}$ has an exact structure corresponding to $\mathbb{A}$ in Theorem 9(4). Then a kernel-cokernel pair $X \to Y \to Z$ in $\mathcal{E}$ which corresponds to a dotted arrow $X \dashleftarrow Z$ in $\mathbb{A}$ is nothing but an *almost split conflation* of $\mathcal{E}$. Furthermore, an indecomposable object X is projective (resp. injective) in the exact category $\mathcal{E}$ if and only if there exists no dotted arrow in $\mathbb{A}$ starting at (resp. ending at) X.

Example 11. Let Λ be a representation-finite algebra and $\mathcal{E} := \mathsf{mod}\,\Lambda$. Then the algebra Γ corresponding to $\mathcal{E}$ is nothing but the classical Auslander algebra for Λ. The translation quiver $Q(\mathcal{E})$ is the same is the usual Auslander-Reiten quiver of $\mathsf{mod}\,\Lambda$. Theorem 9 amounts to say that exact structures on $\mathcal{E}$ are parametrized by (basic) generators, by taking the direct sum of projective objects. This was proved in [3], and actually it is one of the motivations of this study to generalize it.

4. Applications

Now we apply our results to the relative representation theory.

Definition 12. Let Λ be an algebra.

(1) Λ is called an *Iwanaga-Gorenstein* if both $\mathsf{id}\,\Lambda_\Lambda$ and $\mathsf{id}\,{}_\Lambda\Lambda$ are finite.
(2) A module $M \in \mathsf{mod}\,\Lambda$ over an Iwanaga-Gorenstein algebra Λ is called *Cohen-Macaulay* if $\mathrm{Ext}^{>0}_\Lambda(M, \Lambda) = 0$. We denote by $\mathsf{CM}\,\Lambda$ the category of Cohen-Macaulay Λ-modules.
(3) An Iwanaga-Gorenstein algebra Λ is *CM-finite* if $\mathsf{CM}\,\Lambda$ is of finite type.

Example 13. Typical examples of CM-finite Iwanaga-Gorenstein algebras are the following.

- A representation-finite self-injective algebra Λ. In this case, $\mathsf{CM}\,\Lambda = \mathsf{mod}\,\Lambda$ holds.
- An algebra Λ with finite global dimension. In this case, $\mathsf{CM}\,\Lambda = \mathsf{proj}\,\Lambda$ holds.

Note that $\mathsf{CM}\,\Lambda$ is closed under extensions in $\mathsf{mod}\,\Lambda$, thus has the natural exact structure. Concerning Problem (A) in the introduction, we have the following result.

Theorem 14. *Assume* $(*)$ *and suppose that* $\mathcal{E}$ *is an exact category. Then the following are equivalent.*

(1) *There exists an Iwanaga-Gorenstein algebras* Λ*(which is automatically CM-finite) such that* $\mathcal{E}$ *is exact-equivalent to* $\mathsf{CM}\,\Lambda$.
(2) Γ *has finite global dimension, and the classes of projective and injective objects in* $\mathcal{E}$ *coincide.*

Combining this with Theorem 9, we immediately obtain the following classification result.

Corollary 15. *There exists a bijection between the following:*

(1) *Morita-equivalence classes of CM-finite Iwanaga-Gorenstein algebras* Λ.
(2) *Equivalence classes of pairs* $(\Gamma, \mathbb{A})$*, where* Γ *is an algebra with finite global dimension, and* $\mathbb{A}$ *is a set of dotted arrows in* $Q(\mathsf{proj}\,\Gamma)$ *which is a union of oriented cycles.*

For $(\Gamma, \mathbb{A})$ *in* (2)*, the corresponding algebra* Λ *is obtained by taking the endomorphism ring of the direct sum of modules not lying in* $\mathbb{A}$*. In this case,* $\mathsf{CM}\,\Lambda \simeq \mathsf{proj}\,\Gamma$ *holds.*

In this theorem, roughly speaking, Γ parametrizes all possible CM categories $\mathcal{E}$, and for each Γ, $\mathbb{A}$ parametrizes all Iwanaga-Gorenstein algebras whose CM category is $\mathcal{E}$.

Remark 16. Actually, Corollary 15 is a special version of the result about cotilting Λ-modules U such that its Ext-perpendicular category ${}^{\perp}U$ is of finite type, see [6]. In this case, we consider the pair (Λ, U) in (1) and the restriction of $\mathbb{A}$ in (2) is dropped.

Example 17. Put $\Lambda := k[X]/(X^4)$ and $\mathcal{E} := \mathsf{mod}\,\Lambda$. It is well-known that Λ is representation-finite, thus $\mathcal{E}$ is of finite type and the corresponding Γ is the classical Auslander algebra of Λ. Then $Q(\mathcal{E})$ looks like

$$1 \leftrightarrows 2 \leftrightarrows 3 \leftrightarrows 4$$

together with three dotted loops starting at 1, 2 and 3. Then there exists $2^3 = 8$ choices of $\mathbb{A}$ satisfying Corollary 15 since every set of dotted arrows are allowed. The resulting eight Iwanaga-Gorenstein algebras are all the Iwanaga-Gorenstein algebras such that their CM category is equivalent to $\mathcal{E}$. All of this are 2-Iwanaga-Gorenstein. More generally, starting from representation-finite algebra Λ, this procedure yields CM-finite 2-Iwanaga-Gorenstein algebras. This process is the same as τ-selfinjective algebras in [2], or 0-precluster-tilted algebras in [8].

Corollary 15 actually gives somewhat computable algorithm to produce CM-finite Iwanaga-Gorenstein algebras. More precisely, all CM-finite Iwanaga-Gorenstein algebras Λ are obtained by the following steps.

(1) Take an algebra Γ with finite global dimension.
(2) Compute $Q(\mathsf{proj}\,\Gamma)$
(3) For each cyclic orbit $\mathbb{A}$ of $Q(\mathsf{proj}\,\Gamma)$, compute the endomorphism ring Λ of vertices not lying on $\mathbb{A}$.

Moreover, Corollary 15 seems to be the best possible classification of CM-finite Iwanaga-Gorenstein algebras by the following reason: If an algebra Λ has finite global dimension, then Λ is Iwanaga-Gorenstein with $\mathsf{CM}\,\Lambda = \mathsf{proj}\,\Lambda$, hence is CM-finite. Therefore the complete classification of CM-finite Iwanaga-Gorenstein algebras, if exists, should contain that of algebras with finite global dimension, which is unlikely to be settled without any further restriction. Corollary 15 decomposes the classification problem of CM-finite algebras into (1) that of algebras with finite global dimension and (2) the computation of the translation quiver associated with the algebra.

However, there are several problems on Corollary 15. One of those is that, for a given algebra Γ, there seems to be no systematic algorithm to draw $Q(\mathsf{proj}\,\Gamma)$, without calculating explicitly by hand. For actual computation, we first should chose algebras with finite global dimension, but there are too many and it is not clear which classes of such algebras are suitable for computation.

One of the candidates is the class of *strict* τ*-algebra* introduced in [7], which is an Auslander algebra of torsion classes of finite type. Purely combinatorial conditions on the translation quiver Q are known such that its mesh algebra $\Gamma := k(Q)$ yields a strict τ-algebra, and in this case $Q(\mathsf{proj}\,\Gamma) = Q$ holds. Thus we do not have to compute the translation quiver $Q(\mathsf{proj}\,\Gamma)$ if we start from Q.

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