DEGENERATIONS OF COHEN-MACAULAY MODULES VIA MATRIX REPRESENTATIONS

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Abstract. We discuss the degeneration problem for Cohen-Macaulay modules via matrix representations. We shall give the description of such degenerations over hypersurfaces of countable Cohen-Macaulay representation type (A_∞^d).

1. Introduction

This report is based on a joint work with Yuji Yoshino.

The notion of degenerations of modules appears in geometric methods of representation theory of finite dimensional algebras. Yoshino [6] gives a scheme-theoretical definition of degenerations, so that it can be considered for modules over a noetherian algebra which is not necessary finite dimensional. Many authors have studied the degeneration problem of modules [7, 14, 15]. The author and Yoshino [3] give the complete description of degenerations over a ring of even-dimensional simple hypersurface singularity of type (A_n).

Let $(R, \mathfrak{m})$ a commutative noetherian complete local k-algebra with a residue field k. It is known that, since R is complete, there exists a regular local k-subalgebra S of R such that R is a module-finite S-algebra. Let M be a Cohen-Macaulay R-module. Then M is free as S-module, so that we can obtain a k-algebra homomorphism $R \to \mathrm{End}_S(M)$. It is called a matrix-representation of M over S.

The purpose of the report is to give the necessary condition for the degenerations of Cohen-Macaulay modules by considering it via matrix representations. As an application, we will give the description of degenerations of indecomposable Cohen-Macaulay modules over hyper surfaces of countable representation type (A_∞) in the case where R is of dimension 1 and 2.

2. matrix representation

Throughout the paper, let k be an algebraically closed field of characteristic zero and $(R, \mathfrak{m})$ a commutative noetherian complete local k-algebra and assume that $k \cong R/\mathfrak{m}$. First we recall a notion of matrix representations of Cohen-Macaulay modules. For the reference, we recommended the reader to refer to [13].

Since R is a complete ring, by Cohen's structure theorem, there exists a regular local k-subalgebra S of R such that R is a module-finite S-algebra. One can show that S is isomorphic to a formal power series ring over k.

This is not in final form. The detailed version will be submitted to elsewhere for publication.

Definition 1. We say that R is a Cohen-Macaulay ring if R is free as an S-module. We also say that a finitely generated R-module M is Cohen-Macaulay if M is free as an S-module. We denote by $\mathrm{CM}(R)$ the category of all Cohen-Macaulay R-modules and all R-homomorphisms.

Given a Cohen-Macaulay R-module M, since M is isomorphic to S^n for some $n \geq 0$, we have a k-algebra homomorphism

$$R \to \mathrm{End}_S(M) \cong S^{n \times n},$$

which is a matrix representation of R over S.

Example 2. Let $R = k[[x,y]]/(x^2)$. It is known that R is of countable representation type and isomorphis classes of indecomposable Cohen-Macaulay modules are the following:

$$R, \quad R/(x), \quad (x, y^n) \quad n \geq 1.$$

Then the matrix representations of these modules are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (0), \quad \begin{pmatrix} 0 & y^n \\ 0 & 0 \end{pmatrix} \quad n \geq 1.$$

See [9, (6.5)] for example.

Next let us recall the notion of degenerations of finitely generated R-modules. See [10, 11, 12] for the details.

Definition 3. Let R be a noetherian algebra over a field k, and let M and N be finitely generated left R-modules. We say that M degenerates to N, or N is a degeneration of M, if there is a discrete valuation ring (V, tV, k) that is a k-algebra (where t is a prime element) and a finitely generated left $R \otimes_k V$-module Q which satisfies the following conditions:

(1) Q is flat as a V-module.
(2) $Q/tQ \cong N$ as a left R-module.
(3) $Q[1/t] \cong M \otimes_k V[1/t]$ as a left $R \otimes_k V[1/t]$-module.

Remark 4. Let M, N and L be finitely generated R-modules.

(1) Suppose that M degenerates to N. Then as a discrete valuation ring V in Definition 3 we can always take the ring $k[t]_{(t)}$. Thus we always take $k[t]_{(t)}$ as V. Moreover, let $T = k[t]\backslash(t)$ and $T' = k[t]\backslash\{0\}$. Then we also have $R \otimes_k V = T^{-1}R[t]$ and $R \otimes_k V_t = T'^{-1}R[t]$. See [11, Corollary 2.4., Remark 3.1.].
(2) Assume that there is an exact sequence of finitely generated R-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0.$$

Then M degenerates to $L \oplus N$. See [11, Remark 2.5] for the detail.
(3) Suppose that M degenerates to N. Then the ith Fitting ideal of M contains that of N for all $i \geq 0$. Namely, denoting the ith Fitting ideal of an R-module M by $\mathcal{F}_i^R(M)$, we have $\mathcal{F}_i^R(M) \supseteq \mathcal{F}_i^R(N)$ for all $i \geqq 0$. (See [12, Theorem 2.5]).

Proposition 5. *Let M and N be Cohen-Macaulay R-modules. Suppose that M degenerates to N. Let Q be a finitely generated $R \otimes_k V$-module which gives the degeneration. Then Q is free as an $S \otimes_k V$-module.* □

Proof. Since $V = T^{-1}k[t]$ where $T = k[t]\backslash(t)$ (Remark 4 (1)), we can take a finitely generated $R[t]$-submodule Q' of Q such that $T^{-1}Q' = Q$. Then Q' is flat over $k[t]$, $Q'_0 \cong N$ and $Q'_c \cong M$ for each $c \in k^*$. Here Q'_c is defined to be $Q'/(t-c)Q'$ for an element $c \in k$. See [11, Theorem 3.2.]. Then we can show that Q' is projective as an $S[t]$-module. Note that each projective $S[t]$-module is $S[t]$-free by the fact on the Bass-Quillen conjecture [5, Chapter V Theorem 5.1]. Hence, Q' is $S[t]$-free, so that Q is free as an $S \otimes_k V$-module.. □

Let M and N be Cohen-Macaulay R-modules and suppose that M degenerates to N. Then there exits a finitely generated $R \otimes_k V$-module Q which satisfies the definition of the degeneration. By virtue of Proposition 5, Q is free as an $S \otimes_k V$-module. Thus we can consider the matrix representation of Q over $S \otimes_k V$. For matrices μ and ν, we denote by $\mu \cong \nu$ if there exists an invertible matrix P such that $P^{-1}\mu P = \nu$.

Corollary 6. *Let $R = S/(f)$ be a hypersurface ring and M and N Cohen-Macaulay R-modules. Then M degenerates to N if and only if there exists the matrix representation ξ over $S \otimes_k V$ such that $\xi \otimes V/t \cong \nu$ and $\xi \otimes V_t \cong \mu \otimes V_t$ hold, where ν and μ are the matrix representation of N and M over $S \otimes_k V$ respectively.* □

For a matrix representation μ over S, we denote by $I_j(\mu)$ the ideal of S generated by j-minors of μ and also denote by $\mathrm{tr}(\mu)$ the trace of μ.

Corollary 7. *Let Q be a free $S \otimes_k V$-module and M be a Cohen-Macaulay R-module. Suppose that Q_t is isomorphic to $M \otimes_k V_t$. We denote by μ (resp. ξ) the matrix representation of M (resp. Q) over S. Then we have the following equalities:*

(1) $\mathrm{tr}(\xi) = \mathrm{tr}(\mu)$,
(2) $\det(\xi) = \det(\mu)$,
(3) *For all $j \geq 0$, there exist l such that $I_j(\xi) = t^l I_j(\mu)$.*

Here, (1) and (2) may be equalities in S and (3) may be a equality in $S \otimes_k V$. □

Taking Corollary 6 into consideration, Corollary 7 gives the necessary condition of the degeneration. In the next section, we shall apply the condition to the case where the base ring is of type (A_∞^d) with $d = 1, 2$.

3. Degenerations of Cohen-Macaulay modules over (A_∞)

Let $R = k[[x_0, x_1, x_2, \cdots, x_d]]/(f)$ where f is of the form:

$$f = x_0^2 + x_2^2 + \cdots + x_d^2.$$

Then R has a countable Cohen-Macaulay representation type (A_∞^d). We say that R has a countable Cohen-Macaulay representation type if there are only countably many isomorphism classes of maximal Cohen-Macaulay modules. The indecomposable Cohen-Macaulay R-modules are classified (cf. [8]). In this section, we shall describe the degenerations of indecomposable Cohen-Macaulay R-modules when $d = 1, 2$.

Suppose that $\dim R = 1$. As mentioned in Example 2, matrix representations of the indecomposable Cohen-Macaulay R-modules are the following:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (0), \quad \begin{pmatrix} 0 & y^n \\ 0 & 0 \end{pmatrix} \quad n \geq 1.$$

Theorem 8. *Let $R = k[[x,y]]/(x^2)$. Then $\begin{pmatrix} 0 & y^a \\ 0 & 0 \end{pmatrix}$ degenerates to $\begin{pmatrix} 0 & y^b \\ 0 & 0 \end{pmatrix}$ if and only if $a \le b$ and $a \equiv b \mod 2$.*

Proof. First we notice that $a \le b$ if $\begin{pmatrix} 0 & y^a \\ 0 & 0 \end{pmatrix}$ degenerates to $\begin{pmatrix} 0 & y^b \\ 0 & 0 \end{pmatrix}$ by Remark 4(3).

Suppose that $a \equiv b \mod 2$. We consider the matrices ξ as the matrix representation of Q:

$$\xi = \begin{pmatrix} ty^{\frac{a+b}{2}} & y^b \\ -t^2 y^a & -ty^{\frac{a+b}{2}} \end{pmatrix}.$$

Then one can show that Q gives the degeneration from $\begin{pmatrix} 0 & y^a \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & y^b \\ 0 & 0 \end{pmatrix}$.

For the converse, we only prove the following case (the general case can be proved by reducing to the case of the claim).

Claim: R never degenerates to $\begin{pmatrix} 0 & y^{2m+1} \\ 0 & 0 \end{pmatrix}$.

Suppose that R degenerates to (x, y^{2m+1}). Since the matrix representations of R and (x, y^{2m+1}) over S are $\mu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\nu = \begin{pmatrix} 0 & y^{2m+1} \\ 0 & 0 \end{pmatrix}$ respectively, after applying elementary transformation, we may assume that the matrix representation of Q over $S \otimes_k V$ which gives the degeneration is of the form:

$$\xi = \begin{pmatrix} 0 & y^{2m+1} \\ 0 & 0 \end{pmatrix} + t \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} t\alpha & y^{2m+1} + \beta \\ t\gamma & t\delta \end{pmatrix}.$$

Then $\xi \otimes V_t \cong \mu \otimes V_t$. By Corollary 7, we have

$$t\delta = -t\alpha$$

since $\operatorname{tr}(\xi) = 0$. Moreover,

$$\det \xi = -t^2\alpha^2 - t\gamma(y^{2m+1} + t\beta) = 0, \tag{3.1}$$

$$I_1(\xi) = (t\alpha, t\gamma, y^{2m+1} + t\beta) \supseteq (t^l) \text{ for some } l. \tag{3.2}$$

Note that the above equalities are obtained in $S \otimes_k V$. From the equation (3.1), we have

$$t\alpha^2 = \gamma(y^{2m+1} + t\beta)$$

in $S \otimes_k V$. Since t does not divide $y^{2m+1} + t\beta$, t divides γ, so that $\gamma = t\gamma'$ for some $\gamma' \in S \otimes_k V$. Hence we also have the equality in $S \otimes_k V$:

$$\alpha^2 = \gamma'(y^{2m+1} + t\beta). \tag{3.3}$$

Since S is a UFD, so is $S[t]$. Thus $S \otimes_k V = T^{-1}S[t]$ is also a UFD. Take the unique factorization into prime elements of $y^{2m+1} + t\beta$:

$$y^{2m+1} + t\beta = P_1^{e_1} P_2^{e_2} \cdots P_n^{e_n}.$$

Since the equation (3.3) holds, there exists i such that e_i is an odd number. Then P_i divides α, so that P_i also divides γ'. Therefore,

$$(P_i) \supseteq I_i(\xi) \supseteq (t^l),$$

so that $P_i = t$. This makes contradiction since t cannot divide $y^{2m+1} + t\beta$. $\square$

Next we consider the case when $R = k[[x, y, z]]/(x^2 + y^2)$. Replacing X (resp. Y) with $x + iy$ (resp. $2iy$), we may consider $k[[X, Y, z]]/(X^2 - XY)$ as R. We rewrite X and Y by x and y respectively. Then one can see that all indecomposable Cohen-Macaulay R-modules are R and ideals of the form;

$$(x), \quad (x - y), \quad (x, z^n), \quad (x - y, z^n), \quad n \geq 1.$$

See [1, Proposition 2.2] for example. One can also show that the matrix representations over $S = k[[y, z]]$ of the modules are

$$(y), \quad (0), \quad \begin{pmatrix} y & z^n \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & z^n \\ 0 & y \end{pmatrix}, \quad n \geq 1$$

respectively. For instance, let M be the ideal (x, z^n). Then M has a basis x and z^n as an S-module, that is, $M \cong xS \oplus z^nS$. Note that the matrix representation of M is obtain from the action of x on M. The multiplication map a_x by x on M induces the correspondence:

$$a_x : xS \oplus z^nS \to xS \oplus z^nS; \quad \begin{pmatrix} ax \\ bz^n \end{pmatrix} \mapsto \begin{pmatrix} ayx + bz^nx \\ 0 \end{pmatrix}.$$

Hence the matrix representation of M is $\begin{pmatrix} y & z^n \\ 0 & 0 \end{pmatrix}$.

We state the description of the degenerations without the proof.

Theorem 9. *Let $R = k[[x, y, z]]/(x^2 - yx)$. Then $\begin{pmatrix} 0 & z^a \\ 0 & y \end{pmatrix}$ (resp. $\begin{pmatrix} y & z^a \\ 0 & 0 \end{pmatrix}$) never degenerates to $\begin{pmatrix} 0 & z^b \\ 0 & y \end{pmatrix}$ and $\begin{pmatrix} y & z^b \\ 0 & 0 \end{pmatrix}$ for all $a < b$.* $\square$

Remark 10. Araya, et al.[1] show that the Cohen-Macaulay modules which appear in Theorem 3.3 (resp. Theorem 9) are obtain from the extension by $R/(x)$ and itself (resp. $R/(x)$ and $R/(y)$). And in the case, we have the degeneration by Remark 4(2). Summing up this fact, we obtain the description of the degenerations of indecomposable Cohen-Macaulay modules over $k[[x, y]]/(x^2)$ and $k[[x, y, z]]/(x^2 + xy)$.

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