THE DEFINING RELATIONS AND THE CALABI-YAU PROPERTY OF 3-DIMENSIONAL QUADRATIC AS-REGULAR ALGEBRAS

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Abstract. In this report, we consider 3-dimensional quadratic AS-regular algebras. By using the normalization of a variety, we determine the defining relations of Type CC and Type NC 3-dimensional quadratic AS-regular algebras (these algebras correspond to cuspidal and nodal cubic curves in the projective plane). Also, we consider the following conjecture: for a 3-dimensional quadratic AS-regular algebra A, there exists a Calabi-Yau AS-regular algebra C such that A and C are graded Morita equivalent. Using the twist of a superpotential in the sense of Mori-Smith and the defining relations determined above, we shall conform this conjecture for all cases apart from the case of the elliptic curve.

1. AS-regular algebras and geometric algebras

Through this report, let k be an algebraically closed field of characteristic 0, A a graded k-algebra finitely generated in degree 1. That is, $A = T(V)/I$, where V is a k-vector space, $T(V)$ is the tensor algebra of V and I is a homogeneous two-sided ideal of $T(V)$ with $I_0 = I_1 = 0$.

Artin-Schelter [1] defined *AS-regular algebras*. Moreover, Artin-Tate-Van den Bergh [2] classified AS-regular algebras of global dimension 3 via geometry.

Definition 1. ([1]) Let A be a noetherian connected graded k-algebra. A is called *d-dimensional AS-regular* if A satisfies the following conditions:

(i) $\operatorname{gldim} A = d < \infty$,

(ii) (*Gorenstein condition*) $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & (i = d), \\ 0 & (i \neq d). \end{cases}$

We consider now the case where I is an ideal of $k\langle x_1, \dots, x_n\rangle$ generated by homogeneous polynomials of degree two, that is, the quotient algebra $A = k\langle x_1, \dots, x_n\rangle/I$ is a quadratic algebra. In that case, we set

$$\Gamma_A := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f(p, q) = 0 \text{ for all } f \in I_2\},$$

where, for points $p = (a_1, \dots, a_n)$, $q = (b_1, \dots, b_n) \in \mathbb{P}^{n-1}$ and a homogeneous polynomial $f = \sum_{i,j} \alpha_{i,j} x_i x_j$ of degree 2, we define

$$f(p, q) := \sum_{i,j} \alpha_{i,j} a_i b_j.$$

The notion of a *geometric algebra* over k was introduced in [5].

Definition 2. ([5]) Let $A = k\langle x_1, \dots, x_n\rangle/I$ be a quadratic k-algebra.

The detailed version of this paper will be submitted for publication elsewhere.

(i) A satisfies (G1) if there exists a pair (E, σ) where E is a closed k-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut} E$ such that

$$\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid p \in E\}.$$

In this case, we write $\mathcal{P}(A) = (E, \sigma)$ called *the geometric pair of A*.

(ii) A satisfies (G2) if there exists a pair (E, σ) where E is a closed k-subscheme of $\mathbb{P}^{n-1}$ and $\sigma \in \operatorname{Aut} E$ such that

$$I_2 = \{f \in k\langle x_1, \dots, x_n\rangle_2 \mid f(p, \sigma(p)) = 0, \text{ for all } p \in E\}.$$

In this case, we write $A = \mathcal{A}(E, \sigma)$.

(iii) A is called *geometric* if A satisfies both (G1) and (G2), and $A = \mathcal{A}(\mathcal{P}(A))$.

Note that, if A satisfies (G1), A determines the pair (E, σ) by using Γ_A. Conversely, if A satisfies (G2), A is determined by the pair (E, σ).

The following theorem due to [5] is needed to prove our main results in this report. Classifying geometric algebras is equivalent to classifying geometric pairs in the following sense.

Theorem 3. ([5]) *Let $A = \mathcal{P}(A) = (E, \sigma)$, $A' = \mathcal{P}(A') = (E', \sigma')$ be geometric algebras.*

(1) *$A \cong A'$ if and only if there exists $\tau \in \operatorname{Aut} \mathbb{P}^{n-1}$ that restricts to an isomorphism between E and E' such that the following diagram commutes*:

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\tau"'] & E \arrow[d, "\tau"] \\ E' \arrow[r, "\sigma'"] & E' \end{tikzcd}$$

(2) *A and A' are graded Morita equivalent if and only if there exists a sequence $\{\tau_i\}_{i \in \mathbb{Z}}$ where $\tau_i \in \operatorname{Aut} \mathbb{P}^{n-1}$ restricts to an isomorphism between E and E' for all i and the following diagrams commute*:

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\tau_i"'] & E \arrow[d, "\tau_{i+1}"] \\ E' \arrow[r, "\sigma'"] & E' \end{tikzcd}$$

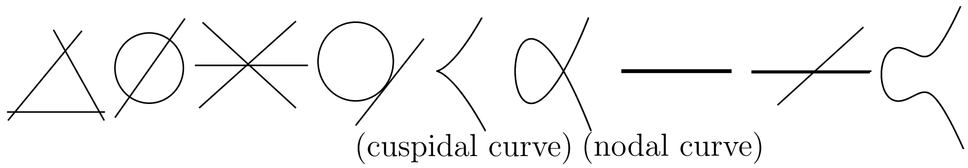
In this report, we consider 3-dimensional quadratic AS-regular algebras. These were classified by Artin-Tate-Van den Bergh [2] using a geometric pair (E, σ).

Theorem 4. ([2]) *Every 3-dimensional quadratic AS-regular algebra A is geometric. Moreover, when $\mathcal{P}(A) = (E, \sigma)$, E is either the projective plane $\mathbb{P}^2$ or a cubic divisor in $\mathbb{P}^2$. where E is a cubic curve of $\mathbb{P}^2$ and σ is an automorphism of E.*

Thus we know that E must be one of the following:

(1) $\mathbb{P}^2$
(2) The union of three lines making a triangle
(3) The union of a line and a conic meeting at two points
(4) The union of three lines meeting at one point
(5) The union of a line and a conic meeting at one point
(6) A cuspidal cubic curve

(7) A nodal cubic curve
(8) A triple line
(9) The union of a double line and a line
(10) An elliptic curve



(cuspidal curve) (nodal curve)

Until now, the 3-dimensional quadratic AS-regular algebras in the cases (1) to (5) were classified up to graded algebra isomorphism and graded Morita equivalence. Cases (6) through (10) proved troublesome as $\sigma \in \operatorname{Aut} E$ is difficult to find in each of these cases. However, it is possible to use the normalization of an algebraic variety to determine the forms of automorphisms of E in cases (6) and (7). This is what we will introduce here. From now we denote by Type CC and Type NC the algebras of cases (6) and (7) respectively.

2. Classifications of Type CC, Type NC Algebras

First we discuss the normalization of a variety.

Let E be an irreducible affine variety. If the coordinate ring $k[E]$ of E is a normal ring, then we say that E is a *normal variety*. Also, let $\tilde{E}$ be a normal variety and $\pi\colon \tilde{E} \to E$ be a surjection. We say that π is a *normalization* of E if $\overline{\pi^*(k[E])} = k[\tilde{E}]$.

In the projective case we take an affine cover and construct a normalization via the gluing of the normalizations from each component of the cover. In order to determine $\sigma \in \operatorname{Aut} E$ we use the following important theorem:

Theorem 5. *Let E be an irreducible variety and $\pi\colon \tilde{E} \to E$ a normalization of E. Then for any $\sigma \in \operatorname{Aut} E$ there exists a unique $\varphi \in \operatorname{Aut} \tilde{E}$ such that $\sigma \circ \pi = \pi \circ \varphi$. I.e, the following diagram commutes*:

$$\begin{tikzcd} \tilde{E} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & E \arrow[d, "\sigma"] \\ \tilde{E} \arrow[r, "\pi"] & E \end{tikzcd}$$

<u>Type CC</u>
Let $E = \mathcal{V}(x^3 - y^2 z)$. Then $\pi\colon \mathbb{P}^1 \to E$ given by

$$\pi(a\colon b) = (a^2 b\colon a^3\colon b^3)$$

is a normalization of E.

<u>Type NC</u>
Let $E = \mathcal{V}(x^3 + y^3 + xyz)$. Then $\pi\colon \mathbb{P}^1 \to E$ given by

$$\pi(a\colon b) = (a^2 b\colon ab^2\colon -a^3 - b^3)$$

is a normalization of E.

For both Type CC and Type NC we can take φ from Theorem 5 to be an automorphism of $\mathbb{P}^1$. In particular we can assume $\varphi \in \mathrm{PGL}_2(k)$. From this we can use our $\varphi \in \operatorname{Aut} \mathbb{P}^1$ to determine our $\sigma \in \operatorname{Aut} E$. These are as follows:

Type CC

$$\sigma_r(x\colon y\colon z) = (rxy + x^2\colon xy\colon r^3xy + 3r^2x^2 + 3ryz + xz) \quad (r \neq 0, 1).$$

Type NC

$$\sigma_{1,s}(x\colon y\colon z) = (sxy\colon s^2y^2\colon (s^3-1)x^2 + s^3yz) \quad (s^3 \neq 0, 1)$$

or

$$\sigma_{2,t}(x\colon y\colon z) = (ty^2\colon t^2xy\colon (1-t^3)x^2 + yz) \quad (t^3 \neq 0, 1).$$

We can use the fact that 3-dimensional quadratic AS-regular algebras are geometric algebras and determine our Type CC, Type NC algebras using the pair (E, σ). We can then use Theorem 3 to classifying these up to graded algebra isomorphism and graded Morita equivalence.

2.1. **Type CC.** From the pair (E, σ_r) we have that Type CC algebras can be written in the form:

$$A = \mathcal{A}(E, \sigma_r) = k\langle x, y, z\rangle / \left(\begin{array}{c} -3r^2x^2 + 2r^3xy + xz - zx - 2rzy, \\ xy - yx + ry^2, \\ -3rx^2 - r^3y^2 + yz - zy \end{array} \right)$$

Theorem 6. *For arbitrary $r, r' \neq 0, 1$, we have that $\mathcal{A}(E, \sigma_r) \cong \mathcal{A}(E, \sigma_{r'})$. In particular, up to graded algebra isomorphism, there is only one Type CC 3-dimensional quadratic AS-regular algebra.*

2.2. **Type NC.** As $\sigma \in \operatorname{Aut} E$ can take 2 different forms we split the classification of Type NC algebras into 2 cases, which is called Type NC_1 when $\sigma = \sigma_{1,s}$, and Type NC_2 when $\sigma = \sigma_{2,t}$, respectively. From the pairs (E, σ) we have that Type NC algebras can be written in one of the following forms.

- Type NC_1 $(\sigma = \sigma_{1,s})$

$$A = \mathcal{A}(E, \sigma_{1,s}) = k\langle x, y, z\rangle / \left(\begin{array}{c} xy - syx, \\ (s^3-1)x^2 + s^2zy - syz, \\ (s^3-1)y^2 + s^2xz - szx \end{array} \right)$$

  In this case we have that for $A = \mathcal{A}(E, \sigma_{1,s})$ and $A' = \mathcal{A}(E, \sigma_{1,s'})$, $A \cong A'$ if and only if $s' = s^{\pm 1}$.

  Also, A and A' are graded Morita equivalent if and only if ${s'}^3 = s^{\pm 3}$.

- Type NC_2 $(\sigma = \sigma_{2,t})$

$$A = \mathcal{A}(E, \sigma_{2,t}) = k\langle x, y, z\rangle / \left(\begin{array}{c} txz + (1-t^3)yx - t^2zy, \\ tzx + (1-t^3)xy - t^2yz, \\ y^2 - tx^2 \end{array} \right)$$

  Also, for arbitrary t, t' we have that $\mathcal{A}(E, \sigma_{2,t}) \cong \mathcal{A}(E, \sigma_{2,t'})$. In particular, up to graded algebra isomorphism, there is only one Type NC_2 3-dimensional quadratic AS-regular algebra.

Theorem 7. *For arbitrary s, t, we have $\mathcal{A}(E, \sigma_{1,s}) \not\cong \mathcal{A}(E, \sigma_{2,t})$. But, for arbitrary t, $\mathcal{A}(E, \sigma_{1,-1})$ and $\mathcal{A}(E, \sigma_{2,t})$ are graded Morita equivalent.*

3. Calabi-Yau algebras and superpotentials

In this section we describe another of the main result of our research. First we recall the definition of a Calabi-Yau algebra.

Definition 8. ([4]) Let A be a connected graded noetherian k-algebra. If A satisfies the following conditions, then A is called *d-dimensional Calabi-Yau*:

(i) $\mathrm{pd}_{A^{\mathrm{e}}} A = d < \infty$,

(ii) $\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \cong \begin{cases} A & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$ (as left Λ^{e}-modules)

where $A^{\mathrm{e}} = A \otimes_k A^{\mathrm{op}}$ is the enveloping algebra of A.

For example, it is known that an n-th polynomial ring $k[x_1, x_2, \ldots, x_n]$ is n-dimensional Calabi-Yau. Also, the preprojective algebra of a path algebra of infinite-type is Calabi-Yau.

Using a geometric pair (E, σ) classified by Theorem 4, we determine the algebras $A = \mathcal{A}(E, \sigma)$. Note that a 3-dimensional quadratic AS-regular algebra A is Koszul, and that the quadratic dual $A^!$ of A is a Frobenius algebra by [8, Proposition 5.10]. Then, we consider the Nakayama automorphism of $A^!$. By using the following theorem due to Reyes-Rogalski-Zhang [7], we can investigate whether or not a 3-dimensional Koszul AS-regular algebra A is Calabi-Yau.

Theorem 9. ([7]) *If A is a 3-dimensional Koszul AS-regular algebra. Then, A is Calabi-Yau if and only if the Nakayama automorphism of $A^!$ is identity (that is, $A^!$ is symmetric).*

Next, we recall the definition of a superpotential:

Definition 10. ([3], [6]) For a finite-dimensional k-vector space V, we define the k-linear map ϕ: $V^{\otimes 3} \longrightarrow V^{\otimes 3}$ by

$$\phi(v_1 \otimes v_2 \otimes v_3) := v_3 \otimes v_1 \otimes v_2.$$

If $\phi(w) = w$ for $w \in V^{\otimes 3}$, then w is called *superpotential*. Also, for $\tau \in \mathrm{GL}(V)$, we define

$$w^\tau := (\tau^2 \otimes \tau \otimes \mathrm{id})(w),$$

where $\mathrm{GL}(V)$ is the general linear group of V.
Moreover, for a finite-dimensional k-vector space V and a subspace W of $V^{\otimes 3}$, we set

- $\partial W := \{(\psi \otimes \mathrm{id}^{\otimes 2})(w) \mid \psi \in V^*, w \in W\}$,
- $\mathcal{D}(W) := T(V)/(\partial W)$.

For $w \in V^{\otimes 3}$, $\mathcal{D}(w) := \mathcal{D}(kw)$ is called the *derivation-quotient algebra* of w.

In this research another our aim is to solve the following two conjectures:

Conjecture For every 3-dimensional quadratic AS-regular algebra A,

(I): there exists a superpotential $w \in V^{\otimes 3}$ and an automorphism τ of V such that A and the derivation-quotient algebra $\mathcal{D}(w^\tau)$ of w^τ are isomorphic as graded algebras;

(II): there exists a Calabi-Yau AS-regular algebra C such that A and C are graded Morita equivalent.

Let A be a graded k-algebra. We denote the category of graded left A-modules by $\mathrm{GrMod}\, A$.

Remark 11. Suppose that Conjecture (I) holds. If $\tau \in \mathrm{GL}\,(V)$ induces $\tau \in \mathrm{Aut}\, \mathcal{D}(w)$, by [6], then $\mathcal{D}(w^\tau) = \mathcal{D}(w)^\tau$. By Conjecture (I) and [9], we have

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, \mathcal{D}(w^\tau) \cong \mathrm{GrMod}\, \mathcal{D}(w)^\tau \cong \mathrm{GrMod}\, \mathcal{D}(w).$$

Since $w \in V^{\otimes 3}$ is a superpotential, $\mathcal{D}(w)$ is a Calabi-Yau algebra. It follows that Conjecture (II) holds.

Consequently, Conjecture (I) implies Conjecture (II).

Our main result is to give partial results for the above two conjectures.

Theorem 12. *For the* 3*-dimensional quadratic AS-regular algebra* $A = \mathcal{A}(E, \sigma)$ *corresponding to* E *and* $\sigma \in \mathrm{Aut}\, E$*, suppose that* E *is* $\mathbb{P}^2$ *or the cubic curve of* $\mathbb{P}^2$ *for all cases apart from the case of the elliptic curve. Then, the conjecture* (I) *and* (II) *hold.*

In particular, we give the proof of Theorem 12 in the cases of Type CC and Type NC.

3.1. **Type CC.** Suppose that (E, σ) is a geometric pair where E is a cuspidal cubic curve in $\mathbb{P}^2$ and $\sigma \in \mathrm{Aut}\, E$. Considering $A = \mathcal{A}(E, \sigma)$ corresponding to (E, σ), $A = k\langle x, y, z\rangle/(f_1, f_2, f_3)$ is 3-dimensional quadratic AS-regular, where

$$\begin{cases} f_1 &= -3r^2x^2 + 2r^3xy + xz - zx - 2rzy, \\ f_2 &= xy - yx + ry^2, \\ f_3 &= -3rx^2 - r^3y^2 + yz - zy. \end{cases}$$

Then, the Koszul dual $A^!$ is $A^! = k\langle X, Y, Z\rangle/(F_1, F_2, F_3, F_4, F_5, F_6)$, where X, Y, Z are the dual k-basis of x, y, z, respectively, and, for $r \neq 0, 1 \in k$,

$$\begin{cases} F_1 &= X^2 + 3r^2XZ + 3rsYZ, \\ F_2 &= YX + XY - 2r^3XZ, \\ F_3 &= Y^2 + rYX + r^3XZ, \\ F_4 &= ZY + 2rsXZ + YZ, \\ F_5 &= ZX + XZ, \\ F_6 &= Z^2. \end{cases}$$

By calculating the Nakayama automorphism of $A^!$,

we have $\nu_{A^!} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore, by Theorem 9, A is Calabi-Yau.

3.2. **Type NC. (i) Type NC$_1$:** Suppose that (E, σ) is a geometric pair where E is a nodal cubic curve in $\mathbb{P}^2$ and $\sigma_{1,s} \in \mathrm{Aut}\, E$. Considering $A_{1,s} = \mathcal{A}(E, \sigma_{1,s})$ corresponding to $(E, \sigma_{1,s})$, $A_{1,s} = k\langle x, y, z\rangle/(f_1, f_2, f_3)$ is a 3-dimensional quadratic AS-regular, where

$$\begin{cases} f_1 &= xy - syx, \\ f_2 &= (s^3 - 1)x^2 + s^2zy - syz, \\ f_3 &= (s^3 - 1)y^2 + s^2xz - szx \qquad (s^3 \neq 0, 1). \end{cases}$$

By calculating the Nakayama automorphism of $A^!_{1,s}$, we have $\nu_{A^!_{1,s}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore, by Theorem 9, $A_{1,s}$ is Calabi-Yau.

(ii) Type NC$_2$: Let

$$w_1 := xyz + yzx + zxy + xzy + zyx + yxz - 2x^3 - 2y^3 \in V^{\otimes 3}.$$

Then, $\phi(w_1) = w_1$, where ϕ is a k-linear map defined in Definition 10. So we see that w_1 is a superpotential. By calculations the derivation quotient algebra $\mathcal{D}(w_1)$ of w_1, $\mathcal{D}(w_1)$ is isomorphic to $A_{1,-1}$, where $A_{1,-1} = \mathcal{A}(E, \sigma_{1,-1})$ is the Type NC$_1$ algebra putting $s = -1$. Note that $\mathcal{D}(w_1)$ is Calabi-Yau because w_1 is a superpotential. Also, let

$$w_2 := -x^2z + 2xyx - xzy - yzx + 2yxy - y^2z - y^2z - zy^2 - zx^2 \in V^{\otimes 3}.$$

By calculations the derivation quotient algebra $\mathcal{D}(w_2)$ of w_2, $\mathcal{D}(w_2)$ is isomorphic to $A_{2,-1}$, where $A_{2,-1} = \mathcal{A}(E, \sigma_{2,-1})$ is the Type NC$_2$ algebra putting $t = -1$. Also we take $\tau := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(V)$. Then

$$w_1^\tau = (\tau^2 \otimes \tau \otimes \mathrm{id})(w) = -w_2.$$

It follows that $A_{2,-1} \cong \mathcal{D}(w^2)$ is isomorphic to $\mathcal{D}(w_1^\tau)$, that is, Conjecture (I) holds.

Moreover, since $\tau(w_1) = (\tau \otimes \mathrm{id} \otimes \mathrm{id})(w_1) = w_1$, we see that $\tau \in \mathrm{GL}(V)$ induces $\tau \in \operatorname{Aut} \mathcal{D}(w_1)$. Then, by Remark 11,

$$\operatorname{GrMod} A_{2,-1} \cong \operatorname{GrMod} \mathcal{D}(w_1^\tau) \cong \operatorname{GrMod} \mathcal{D}(w_1)^\tau \cong \operatorname{GrMod} \mathcal{D}(w_1).$$

Recall that $\mathcal{D}(w_1)$ is Calabi-Yau because w_1 is a superpotential. Therefore, Conjecture (II) holds.

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