# SYMMETRIC HOCHSCHILD EXTENSION ALGEBRAS AND NORMALIZED 2-COCYCLES

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Abstract. For finite dimensional, basic and connected algebras over a field, we give a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric. For bound quiver algebras and arbitrary 2-cocycles we define the normalized 2-cocycle associated with a complete set of orthogonal idempotents, and we show that for every 2-cocycle there exists a normalized 2-cocycle such that their cohomology classes coincide.

## 1. Introduction

Hochschild extensions of algebras give many self-injective algebras. For a finite dimensional algebra $A$ over a field $K$, the trivial extension algebra $T(A) := A \ltimes \mathrm{Hom}_K(A, K)$ of a $K$-algebra $A$ by the standard duality module $\mathrm{Hom}_K(A, K)$ is very important in the representation theory of self-injective algebras. This is also one of the Hochschild extension algebras of $A$. In particular, trivial extension algebras correspond to the zero cocycle in the second Hochschild cohomology groups $\mathrm{H}^2(A, \mathrm{Hom}_K(A, K))$. It is well known that the trivial extension algebra $T(A)$ of $K$-algebra $A$ is symmetric by the symmetric regular $K$-linear map $\mu : T(A) \to K, \mu(a, f) = f(1)$, where $a \in A$ and $f \in \mathrm{Hom}_K(A, K)$. However, it is known that Hochschild extension algebras by duality bimodules are always self-injective [2] but they are not symmetric in general [1].

This paper has two aims:

(1) We will give a sufficient condition, related to 2-cocycles, for Hochschild extension algebras to be symmetric.
(2) For any 2-cocycle $\alpha$ we define normalized 2-cocycles related to a complete set of primitive orthogonal idempotents and construct a 2-cocycle whose cohomology class coincides with the cohomology class of $\alpha$.

## 2. Symmetric Hochschild extension algebras

Let $K$ be a field and $A$ a finite dimensional $K$-algebra. In this section, we recall the definition, the notation and several properties of Hochschild extensions of a $K$-algebra $A$ by a duality bimodule and we give a sufficient condition related 2-cocycles for Hochschild extension algebras to be symmetric.

Let $D$ be a duality between $A$-mod and $A^{\mathrm{op}}$-mod. Then, there is an $A$-bimodule $M$ such that $D \cong \mathrm{Hom}_A(-, M)$. In particular, $M \cong DA$ as $A$-bimodules. Such a module $DA$ is called a duality module. An extension of an algebra $A$ is an epimorphism $\rho : T \to A$ of

The detailed version of this paper will be submitted for publication elsewhere.

The author was supported by JSPS Grant-in-Aid for Young Scientists (B) 17K14175.

$K$-algebra. An extension of an algebra $A$ with kernel $DA$ is called a *Hochschild extension* of $A$ by duality module $DA$ if the kernel of $\rho$ is isomorphic to a duality module $DA$ as $T$-bimodule, that is, there exists an exact sequence $0 \to DA \to T \xrightarrow{\rho} A \to 0$. Then, $T$ is called a *Hochschild extension algebra* of $A$ by $DA$.

The Hochschild extension algebra $T$ is defined by a 2-cocycle. A 2-*cocycle* $\alpha : A \times A \to$ is a $K$-bilinear map with the 2-cocycle condition

$$(a, b, c)_\alpha := a\alpha(b, c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)c = 0$$

for $a, b, c \in A$. The Hochschild extension algebra $T \cong A \oplus DA$ as $K$-modules and the multiplication is defined by

$$(a, f)(b, g) = (ab, ag + fb + \alpha(a, b))$$

for $a, b \in A$ and $f, g \in \mathrm{Hom}_K(A, DA)$. We denote such a Hochschild extension algebra $T$ by $T_\alpha(A, DA)$. In particular, the trivial extension of $A$ by $DA$ is the Hochschild extension $T_0(A, DA)$ of $A$ by $DA$ for zero-map.

Hochschild extension algebras of $A$ are related to the second Hochschild cohomology $\mathrm{H}^2(A, DA)$ of $A$ with coefficient in $DA$, which is the cohomology of the complex

$$\mathrm{Hom}_K(A, DA) \xrightarrow{\delta^1} \mathrm{Hom}_K(A^{\otimes 2}, DA) \xrightarrow{\delta^2} \mathrm{Hom}_K(A^{\otimes 3}, DA),$$

where $\delta^1$ and $\delta^2$ are given by

$$[\delta^1(f)](a \otimes b) = af(b) - f(ab) + f(a)b$$
$$[\delta^2(\alpha)](a \otimes b \otimes c) = a\alpha(b \otimes c) - \alpha(ab \otimes c) + \alpha(a \otimes bc) - \alpha(a \otimes b)c$$

for $a, b, c \in A$, $f \in \mathrm{Hom}_K(A, DA)$ and $\alpha \in \mathrm{Hom}_K(A^{\otimes 2}, DA)$. Hochschild extensions $(T) : 0 \to DA \to T \to A \to 0$ and $(T') : 0 \to DA \to T' \to A \to 0$ are called equivalent if there exists a homomorphism $\iota : T \to T'$ as $K$-algebras such that the following diagram commute:

$$\begin{tikzcd} 0 \arrow[r] & DA \arrow[r] \arrow[d, "1"] & T \arrow[r] \arrow[d, "\iota"] & A \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & DA \arrow[r] & T' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

In particular, if Hochschild extension algebras $T, T'$ are equivalent, then $T \cong T'$ as $K$-algebras. It is well known that there exists a one-to-one correspondence between the set of all equivalent classes of Hochschild extensions of $A$ by $DA$ and $\mathrm{H}^2(A, DA)$.

The $K$-linear map $\alpha : A^{\otimes 2} \to DA$ which belongs to $\mathrm{Z}^2(A, DA)$ is induced by a 2-cocycle, so we also call the $K$-linear map $\alpha$ 2-cocycle if there is no confusion. For $f \in \mathrm{Hom}_K(A, DA)$ we define a 2-cocycle $\delta(f)$ by

$$[\delta(f)](a, b) = af(b) - f(ab) + f(a)b.$$

Then for a 2-cocycle $\alpha : A \times A \to DA$, for any $f \in \mathrm{Hom}_K(A, DA)$ the Hochschild extension of $A$ by $DA$ for $\alpha$ and the one for $\alpha - \delta(f)$ are equivalent. In particular, their Hochschild extension algebras are isomorphic.

Let $Q$ be a finite quiver and $A = KQ/I$, where $I$ is an admissible ideal. We denote by $Q_0$ and $Q_1$ the set of all vertices in $Q$, the set of all arrows in $Q$, respectively. Let $Q_0 = \{1, 2, \ldots, n\}$ and $e_i$ the primitive idempotent corresponding to $i \in Q_0$. Then it is well known that $\{e_i \mid i \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $A$.

For a nonzero element $a \in A$, with $a = e_i a e_j$ for some $i, j$, we denote $e_i$ and $e_j$ by $s(a)$ and $t(a)$, respectively. For a path $p$ in $KQ$ we denote by $p$ again the image of $p$ under the canonical map $KQ \to A$ if there is no confusion.

Let $K$ be a field and $A = KQ/I$ a bound quiver algebra. The algebra $A$ is called symmetric if $A$ is isomorphic to $\mathrm{Hom}_K(A, K)$ as $A$-bimodules or, equivalently, there exists a $K$-bilinear map $\mu : A \to K$ such that the following holds:

(S1) $\mu$ is regular, that is, $\mu(Ax) \neq 0$ for any $x \in A$.

(S2) $\mu$ is symmetric, that is, $\mu(xy) = \mu(yx)$ for any $x, y \in A$.

In [2], it is showed that every Hochschild extension algebra $T$ of $A$ by duality module $DA$ is self-injective. In particular, the Nakayama permutation of $T$ and the Nakayama permutation by ${}_A(DA)_A$ coincide. However, Hochschild extension algebras are not symmetric in general. It is shown that there is a Hochschild extension algebra which is symmetric if and only if $DA \cong \mathrm{Hom}_K(A, K)$ by [3, Proposition 2.2]. Thus, we denote by $DA$ the standard duality module $\mathrm{Hom}_K(A, K)$ again. In particular, Hochschild extension algebras by the standard duality module are always weakly symmetric, that is, their Nakayama permutations are identity.

In order to describe our assertion, we explain some notation. For a 2-cocycle $\alpha : A \times A \to DA$, we denote by $\eta_\alpha$ a $K$-bilinear map $A \times A \to DA$ given by $\eta_\alpha(x, y) = \alpha(x, y) - \alpha(y, x)$, where $x, y \in A$. Let $V_\alpha = \{a \in \mathrm{Z}(A) \mid f(a) = 0 \text{ for any } f \in \eta_\alpha(A \times A)\}$.

**Theorem 1.** *If there exists $x_0 \in V_\alpha$ such that $e_i^*(x_0) \neq 0$ for all $i (1 \leq i \leq n)$, then the Hochschild extension algebra $T_\alpha(A, DA)$ of $A$ defined by $\alpha$ is symmetric.*

**Example 2** ([1])**.** Let $Q$ be a quiver with a vertice and three loops $x, y, z$. Let $A = KQ/R_Q^2$, $\mathcal{B} = \{1, x, y, z\}$ a basis of $A$ and $\alpha : A \times A \to DA$ a 2-cocycle given by $\alpha(x, y) = 1^* - z^*$, $\alpha(y, z) = 1^* - x^*$, $\alpha(z, x) = 1^* - y^*$, $\alpha(a, b) = 0$ for $(a, b) \in \mathcal{B} \times \mathcal{B} \setminus \{(x, y), (y, z), (z, x)\}$, where $R_Q$ is the arrow ideal of $KQ$. Then, by direct computation, we have $V_\alpha = \langle 1 + x + y + z \rangle_K$. Since $1^*(1 + x + y + z) = 1$, the Hochschild extension algebra $T_\alpha(A, DA)$ of $A$ by $DA$ for $\alpha$ is symmetric by Theorem 1.

## 3. Normalized 2-cocycles and their applications

Let $A$ be a basic Artin algebra over a commutative Artin ring $K$, $E = \{e_1, e_2, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of $A$, $M$ an $A$-bimodule and $\alpha : A \times A \to M$ a 2-cocycle. If $\alpha$ satisfies that $\alpha(1, a) = \alpha(a, 1) = 0$ for all $a \in A$, then $\alpha$ is called a *normalized* 2-cocycle. Moreover, for every 2-cocycle $\alpha$, $\alpha - \delta f_\alpha$ is a normalized 2-cocycle whose cohomology class coincides the cohomology class of $\alpha$, where $f_\alpha$ is given by $f_\alpha(a) = \alpha(a, 1)$ for $a \in A$.

In this section, we define $E$-normalized 2-cocycles and we show that for every 2-cocycle there exists an $E$-normalized 2-cocycle such that their cohomology class coincide. By means of that construction of $E$-normalized 2-cocycles, for bound quiver algebras we prove a result by Ohnuki, Takeda and Yamagata in [1] as a corollary of Theorem 1.

**Definition 3.** Let $A$ be a basic Artin algebra over a commutative Artin ring $K$, $E = \{e_1, e_2, \ldots, e_n\}$ a complete set of primitive orthogonal idempotents of $A$, $M$ an $A$-bimodule and $\alpha : A \times A \to M$ a 2-cocycle. If $\alpha$ satisfies $\alpha(e_i, A) = \alpha(A, e_i) = 0$ for all $e_i \in E$, then $\alpha$ is called an *$E$-normalized* 2-cocycle.

*Remark* 4. If $\alpha$ is an $E$-normalized 2-cocycle, then $\alpha$ is a normalized 2-cocycle.

From now on, for every 2-cocycle $\alpha$ we will construct an $E$-normalized 2-cocycle whose cohomology class coincides with the cohomology class of $\alpha$.

We will define some notation. For a 2-cocycle $\alpha : A \times A \to M$, we define $h_{\mathrm{R}}(\alpha) \in \mathrm{Hom}_K(A, M)$ by $[h_{\mathrm{R}}(\alpha)](a) = \sum_{k=1}^{n} \alpha(a, e_k)e_k$ for $a \in A$. Similarly, we define $h_{\mathrm{L}}(\alpha) \in \mathrm{Hom}_K(A, M)$ by $[h_{\mathrm{L}}(\alpha)](a) = \sum_{k=1}^{n} e_k\alpha(e_k, a)$ for $a \in A$. Moreover, we put $H_{\mathrm{R}}(\alpha) = \alpha - \delta(h_{\mathrm{R}}(\alpha))$ and $H_{\mathrm{L}}(\alpha) = \alpha - \delta(h_{\mathrm{L}}(\alpha))$ which belong to $\mathrm{Z}^2(A, M)$.

**Proposition 5.** *The following statements hold*:
(1) $[H_{\mathrm{R}}(\alpha)](A, e_i) = 0$ *for every* $i (1 \leq i \leq n)$.
(2) $[H_{\mathrm{L}}(\alpha)](e_i, A) = 0$ *for every* $i (1 \leq i \leq n)$.
(3) $H_{\mathrm{R}}^2(\alpha) = H_{\mathrm{R}}(\alpha)$.
(4) $H_{\mathrm{L}}^2(\alpha) = H_{\mathrm{L}}(\alpha)$.
(5) $H_{\mathrm{L}}H_{\mathrm{R}}(\alpha) = H_{\mathrm{R}}H_{\mathrm{L}}(\alpha)$.

By Proposition 5, we put $\overline{\alpha} = H_{\mathrm{L}}H_{\mathrm{R}}(\alpha) \in \mathrm{Z}^2(A, DA)$ for every 2-cocycle $\alpha$. Then, by direct computation, we have the following properties.

**Proposition 6.** *For a* 2*-cocycle* $\alpha : A \times A \to M$*, the* 2*-cocycle* $\overline{\alpha}$ *satisfies the following*:
(1) *The cohomology class* $[\overline{\alpha}]$ *of* $\overline{\alpha}$ *coincides with the cohomology class* $[\alpha]$ *of* $\alpha$.
(2) *The* 2*-cocycle* $\overline{\alpha}$ *is an* $E$*-normalized* 2*-cocycle.*
(3) *If* $A$ *is a bound quiver algebra* $KQ/I$ *over a field* $K$*, then*

$$\overline{\alpha}(p, q) = \begin{cases} s(p)\alpha(p, q)t(q) - p\alpha(t(p), s(q))q & \text{if } pq \neq 0 \text{ in } KQ \\ 0 & \text{if } pq = 0 \text{ in } KQ \end{cases}$$

*for all paths* $p, q$ *in* $Q$.

As a corollary of Theorem 1, we have [1, Theorem 2.2] by means of the $E$-normalized 2-cocycles.

**Corollary 7** ([1, Theorem 2.2])**.** *Let* $Q$ *be a finite quiver,* $A = KQ/I$ *a bound quiver algebra and* $\alpha : A \times A \to DA$ *a* 2*-cocycle. If* $\alpha$ *satisfies* $\alpha(p, q)(t(q)) = \alpha(q, p)(t(p))$ *for all paths* $p, q$ *which* $pq$ *is a cycle in* $Q$ *and* $p, q \notin Q_0$*, then the Hochschild extension algebra of* $A$ *for* $\alpha$ *is symmetric.*

## References

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