COMPLEX RINGS, QUATERNION RINGS AND OCTONION RINGS

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Abstract. In [2], for any ring R, complex rings $\boldsymbol{C}(R)$, quaternion rings $\boldsymbol{H}(R)$ and octonion rings $\boldsymbol{O}(R)$ are studied. For the real numbers $\mathbb{R}$, $\boldsymbol{H}(\mathbb{R})$ is the Hamilton's quaternions and $\boldsymbol{O}(\mathbb{R})$ is the Kelly-Graves's octonions. In view of progress of quaternions, generalized quaternion algebras $\left(\frac{a,b}{F}\right)$ are introduced for commutative fields F and nonzero elements $a, b \in F$, and these quaternions have been extensively studied as algebraic number theory. In this paper, we use $\boldsymbol{H}(F; a, b)$ instead of $\left(\frac{a,b}{F}\right)$. For a division ring D and nonzero elements $a, b \in Z(D)$, the center of D, we introduce generalized complex rings $\boldsymbol{C}(D; a)$ and generalized quaternion rings $\boldsymbol{H}(D; a, b)$, and study the structure of these rings. We show that these rings are simple rings if the characteristic of D is not 2, that is, $2 \neq 0$, and study the structure of these simple rings. When $2 = 0$, these rings are local quasi-Frobenius rings.

1. Introduction

In 1843 - 1844, Hamilton discovered the quaternions and Kelly, Graves independently discovered the octonions. These numbers are defined over the real numbers and contain the complex numbers. Through Frobenius, Wedderburn and Noether, these numbers have been studied by many mathematicians. We may say that one of the roots of our ring and representation theory began with these numbers.

In order to define these numbers for any ring R, we consider free right R-modules:

$$\begin{aligned}\boldsymbol{C}(R) &= e_0R \oplus e_1R,\\ \boldsymbol{H}(R) &= e_0R \oplus e_1R \oplus e_2R \oplus e_3R,\\ \boldsymbol{O}(R) &= e_0R \oplus e_1R \oplus \cdots \oplus e_7R.\end{aligned}$$

We define $re_i = e_ir$ for any $r \in R$ and any i $(1 \leq i \leq 7)$, and multiplications for $\{e_i\}_i$ are defined by the following Cayley-Graves multiplication table:

$\times$	$\boldsymbol{e}_0$	$\boldsymbol{e}_1$	$\boldsymbol{e}_2$	$\boldsymbol{e}_3$	$\boldsymbol{e}_4$	$\boldsymbol{e}_5$	$\boldsymbol{e}_6$	$\boldsymbol{e}_7$
$\boldsymbol{e}_0$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
$\boldsymbol{e}_1$	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
$\boldsymbol{e}_2$	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
$\boldsymbol{e}_3$	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
$\boldsymbol{e}_4$	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
$\boldsymbol{e}_5$	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
$\boldsymbol{e}_6$	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
$\boldsymbol{e}_7$	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

The detailed version of this paper will be submitted for publication elsewhere.

Then $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ are rings, and $\boldsymbol{O}(R)$ is a non-associative ring. We call $\boldsymbol{C}(R)$ a complex ring, $\boldsymbol{H}(R)$ a quaternion ring and $\boldsymbol{O}(R)$ an octonion ring. For $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$, we put $1 = e_0,\ i = e_1,\ j = e_2,\ k = e_3$. Then multiplications for $\{i, j, k\}$ are usual forms:

$$i^2 = j^2 = k^2 = -1,\ ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j.$$

In order to study $\boldsymbol{H}(\boldsymbol{H}(R))$, we use $\{\mathbb{i}, \mathbb{j}, \mathbb{k}\}$ instead of $\{i, j, k\}$. Namely,

$$\boldsymbol{H}(R) = R + iR + jR + kR,$$
$$\boldsymbol{H}(\boldsymbol{H}(R)) = \boldsymbol{H}(R) + \mathbb{i}\boldsymbol{H}(R) + \mathbb{j}\boldsymbol{H}(R) + \mathbb{k}\boldsymbol{H}(R).$$

Similarly, for $\boldsymbol{C}(\boldsymbol{H}(R)), \boldsymbol{C}(\boldsymbol{C}(R)), \boldsymbol{H}(\boldsymbol{C}(R))$, we use $\{\mathbb{i}, \mathbb{j}, \mathbb{k}\}$.

In view of progress of quaternion rings, generalized quaternion rings are introduced for commutative fields and these rings have been well studied. For later use, we introduce generalized quaternion rings over any ring R.

Let R be a ring and let a, b be non-zero elements $\in Z(R)$, the center of R. Consider the following free right R-modules:

$$\boldsymbol{C}(D; a) = D \oplus iD, \quad \boldsymbol{H}(D; a, b) = D \oplus iD \oplus jD \oplus kD.$$

For these modules and any $r \in R$, we define

$$ri = ir,\ rj = jr,\ rk = kr$$

and multiplications for $\{i, j, k\}$ as follows:

$$i^2 = a,\ j^2 = b,\ ij = -ji = k.$$

Then we can see the following:

$$k^2 = -ab,\ ik = -ki = ja,\ jk = -kj = -ib.$$

By these multiplications, $\boldsymbol{C}(D; a)$ and $\boldsymbol{H}(D; a, b)$ become rings. In this paper, we say $\boldsymbol{C}(D; a)$ a generalized complex ring and $\boldsymbol{H}(D; a, b)$ a generalized quaternion ring. For a commutative field F, $\boldsymbol{H}(F; a, b)$ is denoted by $\left(\frac{a,b}{F}\right)$ and has been extensively studied in algebraic number theory.

Our main purpose of this paper is to study the structures of $\boldsymbol{C}(D), \boldsymbol{H}(D), \boldsymbol{C}(D; a)$ and $\boldsymbol{H}(D; a, b)$ for a given division ring D. By our results, we can see the difference between $\boldsymbol{H}(D; a, b)$ and $\boldsymbol{H}(F; a, b)$. For $\boldsymbol{H}(F; a, b)$, we refer books Lam [4], Nicholson [5], Pierce [6] and Saito [7].

We use following symbols:

$\mathbb{R}$	the real numbers
$\mathbb{Q}$	the rational numbers
$\mathbb{C}$	the complex numbers
$M_n(R)$	$n \times n$ matrix ring over a ring R
$J(R)$	Jacobson radical of R
$S(R_R)$	Socle of R_R
$\lvert X\rvert$	cardinality of a set X
$Pi(R)$	a complete set of orthogonal primitive idempotents of an artinian ring R
$Z(R)$	the center of a ring R

2. Basic concepts and known results

Let R be a finite dimensional algebra over a field F; put $R = x_1F \oplus x_2F \oplus \cdots \oplus x_nF$. For $u \in R$, there exist (f_{ij}) and (g_{ij}) in $M_n(F)$ such that $u(x_1, \ldots, x_n) = (x_1, \ldots, x_n)(f_{ij})$ and ${}^t(x_1, \ldots, x_n)u = (g_{ij})\,{}^t(x_1, \ldots, x_n)$.

Put $r(u) = (f_{ij})$ and $l(u) = (g_{ij})$. $r(u)$ and $l(u)$ are called the right regular representation and left regular representation of u, respectively. If there exists a regular matrix $P \in M_n(F)$ satisfying, for any $u \in F$,

$$r(u)P = Pl(u)$$

then R is called a Frobenius algebra.

In addition, we shall state on Frobenius rings. Let R be a quasi-Frobenius ring. We arrange $Pi(R)$ as

$$Pi(R) = \{e_{11}, e_{12}, \ldots, e_{1m(1)}, e_{21}, e_{22}, \ldots, e_{2m(2)}, \ldots, e_{n1}, e_{n2}, \ldots, e_{nm(n)}\}$$

where

$$e_{ij}R \cong e_{kl}R \text{ if } k = i,$$
$$e_{ij}R \ncong e_{kl}R \text{ if } k \neq i.$$

Put $e_i = e_{i1}$ for $i = 1, \ldots, n$. There exists a (Nakayama) permutation $(e_{\pi(1)}, \ldots, e_{\pi(n)})$ of $(e_1, \ldots, e_n)$ such that $(e_iR; R\pi(i))$ is an i-pair, that is, $S(e_iR) \cong e_{\pi(i)}R/J(e_{\pi(i)}R)$ and $S(Re_{\pi(i)}) \cong Re_i/J(Re_i)$. R is called a Frobenius ring if $m(i) = m(\pi(i))$ for all i. Of course, Frobenius algebras are Frobenius rings (cf. [3]).

3. Structure of quaternion rings $\boldsymbol{H}(D)$

Recently, Lee-Oshiro showed the following results in [2].

Theorem A. *If R is a Frobenius algebra, then $\boldsymbol{C}(R), \boldsymbol{H}(R)$ and $\boldsymbol{O}(R)$ are Frobenius algebras.*

Theorem B. *If R is a quasi-Frobenius ring, then $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$ are quasi-Frobenius rings.*

It follows from Theorem B that, for a division ring D, $\boldsymbol{C}(D)$ and $\boldsymbol{H}(D)$ are quasi-Frobenius rings. Our motivation of this paper is to study these quasi-Frobenius rings.

Let R be a ring. In order to study the structure of $\boldsymbol{C}(R)$ and $\boldsymbol{H}(R)$, we first observe idempotents and nilpotents in these rings. For $\alpha = a + ib + jc + kd \in \boldsymbol{H}(R)$, we write

$$\alpha^2 = A + iB + jC + kD$$

where $a, b, c, d, A, B, C, D \in R$. Then, by calculation, we see

$$A = a^2 - b^2 - c^2 - d^2, \quad B = ba + ab + cd - dc,$$
$$C = ca + ac + db - bd, \quad D = da + ad + bc - cb.$$

Therefore,

$$\alpha^2 = 0 \iff$$

$$(\#)\begin{cases} a^2 - b^2 - c^2 - d^2 &= 0 \\ ba + ab + cd - dc &= 0 \\ ca + ac + db - bd &= 0 \\ da + ad + bc - cb &= 0. \end{cases}$$

Further,

$$\alpha^2 = \alpha \iff$$

$$(*)\begin{cases} a^2 - b^2 - c^2 - d^2 &= a \\ ba + ab + cd - dc &= b \\ ca + ac + db - bd &= c \\ da + ad + bc - cb &= d. \end{cases}$$

By $(*)$, we obtain

Fact 1. Let F be a field with $2 \neq 0$. Then

$$\alpha^2 = \alpha \iff a = \frac{1}{2} \text{ and } \frac{1}{4} + b^2 + c^2 + d^2 = 0.$$

By $(*)$, we can show the following:

Theorem 1. *Assume* $2 \neq 0$.

(1) $J(\boldsymbol{H}(D)) = 0$ *and* $\boldsymbol{H}(D)$ *is a simple ring.*
(2) $|Pi(\boldsymbol{H}(D))| = 1$ *or* 2 *or* 4.
(3) $|Pi(\boldsymbol{H}(D))| = 1$ *iff* $\boldsymbol{H}(D)$ *is a division ring.*
(4) $|Pi(\boldsymbol{H}(D))| = 2 \implies$ *For any primitive idempotent* $e \in \boldsymbol{H}(D)$,

$$\boldsymbol{H}(D) \cong \begin{pmatrix} e\boldsymbol{H}(D)e & e\boldsymbol{H}(D)e \\ e\boldsymbol{H}(D)e & e\boldsymbol{H}(D)e \end{pmatrix}.$$

(5) $D = F$ *is a commutative field*

$$\implies \quad \boldsymbol{H}(F) \text{ is a division ring} \quad \text{or} \quad \boldsymbol{H}(F) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}.$$

(6) *For a commutative field* F, $|Pi(\boldsymbol{H}(F))| = 4$ *does not occur.*

We shall give a sketch of the proof of (1) in this theorem. In order to show $J(\boldsymbol{H}(D)) = 0$, we may show the following.

Lemma 2. *Let* $\alpha \in \boldsymbol{H}(D)$. *Then*

$$\alpha^2 = (\alpha i)^2 = (\alpha j)^2 = (\alpha k)^2 = 0 \Rightarrow \alpha = 0.$$

Proof. Let $\alpha = a + ib + jc + kd \in \boldsymbol{H}(D)$. By $\alpha^2 = 0$ and $(\#)$,

$$(1) \qquad a^2 - b^2 - c^2 - d^2 = 0.$$

Since $\alpha i = -b + ia + jd - kc$ and $(\alpha i)^2 = 0$,

$$(2) \qquad b^2 - a^2 - d^2 - c^2 = 0.$$

Similarly, by $(\alpha j)^2 = 0$ and $(\alpha k)^2 = 0$,

$$c^2 - d^2 - a^2 - b^2 = 0, \tag{3}$$

$$d^2 - c^2 - b^2 - a^2 = 0. \tag{4}$$

By (1) + (2), $-2c^2 - 2d^2 = 0$ and so $c^2 + d^2 = 0$. Hence, $b^2 = a^2$. Similarly, by (1) + (3) and (1) + (4), we obtain $c^2 = a^2$ and $d^2 = a^2$, therefore $a^2 = b^2 = c^2 = d^2$. Since $a^2 - b^2 - c^2 - d^2 = 0$, $2a^2 = 0$ and hence $a = 0$. Thus $a = b = c = d = 0$ and hence $\alpha = 0$, as required. $\square$

Theorem 3. *Assume* $2 = 0$. *Then*

(1) $Pi(\boldsymbol{H}(D)) = 1$.

(2) $\boldsymbol{C}(D)$ *is a local quasi-Frobenius ring such that*

$$J(\boldsymbol{C}(D)) = S(\boldsymbol{C}(D)) = e\boldsymbol{C}(D)$$

where $e = 1 + i$.

(3) $\boldsymbol{H}(D)$ *is a local quasi-Frobenius ring such that*

$$\begin{aligned} J(\boldsymbol{H}(D)) &= (1+i)\boldsymbol{H}(D) + (1+j)\boldsymbol{H}(D), \\ S(\boldsymbol{H}(D)) &= (1+i+j+k)\boldsymbol{H}(D), \end{aligned}$$

$$\begin{tikzcd} & \boldsymbol{H}(D) \arrow[dl, dash] \arrow[dr, dash] & \\ (1+i)\boldsymbol{H}(D) \arrow[dr, dash] & & (1+j)\boldsymbol{H}(D) \arrow[dl, dash] \\ & (1+i+j+k)\boldsymbol{H}(D) & \end{tikzcd} .$$

Sketch of the proof of (1). Let $e = a + bi + cj + dk$ be an idempotent of $\boldsymbol{H}(D)$. By using $(*)$, we can see that $(a+b+c+d)^2 = a+b+c+d$ and hence $a+b+c+d = 0$ or $a+b+c+d = 1$. Then $a+b+c+d = 0$ does not occur, and $a+b+c+d = 1$ implies $a = 1, b = c = d = 0$. Hence $e = 1$.

Example 4. Let consider Hamilton's quaternion ring

$$D := \boldsymbol{H}(\mathbb{R}) = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R},$$

and

$$\boldsymbol{H}(D) = \boldsymbol{H}(\boldsymbol{H}(\mathbb{R})) = \boldsymbol{H}(\mathbb{R}) \oplus \mathbb{i}\boldsymbol{H}(\mathbb{R}) \oplus \mathbb{j}\boldsymbol{H}(\mathbb{R}) \oplus \mathbb{k}\boldsymbol{H}(\mathbb{R}).$$

Then, $|Pi(\boldsymbol{H}(D))| = 4$. In fact, put

$$\begin{aligned} g_1 &= \frac{1}{4}(1 + \mathbb{i}i + \mathbb{j}j + \mathbb{k}k), \quad g_2 = \frac{1}{4}(1 + \mathbb{i}i - \mathbb{j}j - \mathbb{k}k), \\ g_3 &= \frac{1}{4}(1 - \mathbb{i}i - \mathbb{j}j + \mathbb{k}k), \quad g_4 = \frac{1}{4}(1 - \mathbb{i}i + \mathbb{j}j - \mathbb{k}k). \end{aligned}$$

Then, $\{g_1, g_2, g_3, g_4\}$ are orthogonal primitive idempotents and hence $|Pi(\boldsymbol{H}(D))| = 4$.

Further, for a division ring D with $2 \neq 0$, we can show the following result.

Theorem 5. *If $\boldsymbol{H}(D)$ is a division ring, then*

$$\boldsymbol{H}(\boldsymbol{H}(D)) \cong \begin{pmatrix} D & D & D & D \\ D & D & D & D \\ D & D & D & D \\ D & D & D & D \end{pmatrix}.$$

In particular,

$$\boldsymbol{H}(\boldsymbol{H}(\mathbb{R})) \cong \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} & \mathbb{R} & \mathbb{R} \end{pmatrix}.$$

This example and the above theorem show that $\boldsymbol{H}(D)$ and $\boldsymbol{H}(F)$ are different worlds. When $|Pi(\boldsymbol{H}(D))| = 4$, we can obtain the following unexpected result:

Theorem 6. *Let D be a division ring with $2 \neq 0$. The following conditions are equivalent:*

(1) $|Pi(\boldsymbol{H}(D))| = 4$.
(2) *There exist $p, q, r \in D$ such that $p^2 = -1, q^2 = -1, pq = r = -qp$.*

4. Structure of Complex rings $\boldsymbol{C}(D)$

We show the following fact.

Theorem 7. *For a division ring D, the following are equivalent:*

(1) $x^2 \neq -1$ *for all* $x \in D$.
(2) $\boldsymbol{C}(D) = D \oplus iD$ *is a division ring.*

We give a sketch of the proof.

(1) $\Rightarrow$ (2). As is easily seen $2 \neq 0$. Assume that $\boldsymbol{C}(D)$ is not a division ring. Then there exist primitive idempotents $e, f \in \boldsymbol{C}(D)$ such that $\boldsymbol{C}(D) = e\boldsymbol{C}(D) \oplus f\boldsymbol{C}(D)$. Since $\boldsymbol{C}(D)$ is a 2-dimensional D-space, $e\boldsymbol{C}(D) = eD$, and hence there exists $x \in D$ such that $ei = ex$. Set $e = a + ib$ $(a, b, \in D)$. Then, $ei = -b + ia$ and $ex = ax + ibx$. Hence $-b = ax$ and $bx = a$, and it follows $-b = bx^2$ and $-1 = x^2$, a contradiction.

(2) $\Rightarrow$ (1). Note that $2 \neq 0$, because if $2 = 0$, then $(1 + i)^2 = 0$, and hence $1 + i = 0$, a contradiction. Now, assume that there exists $x \in D$ such that $x^2 = -1$. Then $e = \frac{1}{2}(1+xi)$ is an idempotent. Since D is a division ring, e must be 1, a contradiction.

Remark. The implication (1) $\Rightarrow$ (2) is shown in Chapter 10 in [1]. Its proof we state below is complicated but above proof is a ring theoretic one. In fact, let $x = \alpha + i\beta$ $(\alpha, \beta \in D)$ be a non-zero element of $\boldsymbol{C}(D)$. If $\beta = 0$, then $x^{-1} = \alpha^{-1}$. If $\beta \neq 0$, then

$$(\alpha + i\beta)(\beta^{-1}\alpha - i)\beta^{-1}((\alpha\beta^{-1})^2 + 1)^{-1} = 1,$$
$$((\beta^{-1}\alpha)^2 + 1)^{-1}(\beta^{-1}\alpha - i)\beta^{-1}(\alpha + i\beta) = 1.$$

Hence $x^{-1} = (\beta^{-1}\alpha - i)\beta^{-1}((\alpha\beta^{-1})^2 + 1)^{-1}$.

Theorem 8. *Let D be a division ring with $2 \neq 0$. Assume that there exists $x \in D$ such that $x^2 = -1$. Put $e = \frac{1}{2}(1 + ix)$, $f = 1 - e = \frac{1}{2}(1 - ix)$. Then $\boldsymbol{C}(D) = e\boldsymbol{C}(D) \oplus f\boldsymbol{C}(D)$.*

(1) *If $x \in Z(D)$, then $e \in Z(\boldsymbol{C}(D))$, whence $\boldsymbol{C}(D) = e\boldsymbol{C}(D) \times f\boldsymbol{C}(D)$ (ring direct sum).*

(2) *In the case $x \notin Z(D)$, take $d \in D$ such that $xd \neq dx$. Then by calculation, we see that $edf = \frac{1}{4}(d + xdx + i(xd - dx))$, from which we see $edf \neq 0$. Hence it follows $e\boldsymbol{C}(D) \cong f\boldsymbol{C}(D)$, and hence we obtain*

$$\boldsymbol{C}(D) \cong \begin{pmatrix} e\boldsymbol{C}(D)e & e\boldsymbol{C}(D)e \\ e\boldsymbol{C}(D)e & e\boldsymbol{C}(D)e \end{pmatrix}.$$

5. Structure of $\boldsymbol{H}(D; a, b)$ and $\boldsymbol{C}(D; a)$

Let R be a ring. In order to study the structure of $\boldsymbol{C}(R; a)$ and $\boldsymbol{H}(R; a, b)$, we observe idempotents and nilpotents in these rings as in Section 3.

For $\alpha = x + iy + jz + kw \in \boldsymbol{H}(R; a, b)$, we write

$$\alpha^2 = A + iB + jC + kD$$

where $x, y, z, w, A, B, C, D \in R$. Then, by calculation, we see

$$\begin{aligned} A &= x^2 + y^2a + z^2b - w^2ab, \\ B &= xy + yx - zwb + wzb, \\ C &= xz + zx + ywa - wya, \\ D &= xw + yz - zy + wx. \end{aligned}$$

Therefore,

$$\alpha^2 = 0 \iff$$

$$(\#2) \begin{cases} x^2 + y^2a + z^2b - w^2ab &= 0 \\ xy + yx - zwb + wzb &= 0 \\ xz + zx + ywa - wya &= 0 \\ xw + yz - zy + wx &= 0. \end{cases}$$

Further,

$$\alpha^2 = \alpha \iff$$

$$(*2) \begin{cases} x^2 + y^2a + z^2b - w^2ab &= x \\ xy + yx - zwb + wzb &= y \\ xz + zx + ywa - wya &= z \\ xw + yz - zy + wx &= w. \end{cases}$$

By $(*2)$ above, we obtain:

Fact 2. Let F be a field with $2 \neq 0$. Then

$$\alpha^2 = \alpha \iff x = \frac{1}{2} \text{ and } \frac{1}{4} - y^2a - z^2b + w^2ab = 0.$$

Here we state some results on a generalized quaternion ring $\boldsymbol{H}(D; a, b)$, where D is a division ring with $2 \neq 0$. By $(*2)$ above, we can show the following result which corresponds to Theorem 1 in Section 3.

Theorem 9. (1) *$J(\boldsymbol{H}(D; a, b)) = 0$ and $\boldsymbol{H}(D; a, b)$ is a simple ring.*

(2) $|Pi(\boldsymbol{H}(D;a,b))| = 1$ *or* 2 *or* 4.
(3) $|Pi(\boldsymbol{H}(D;a,b))| = 1$ *iff* $\boldsymbol{H}(D;a,b)$ *is a division ring.*
(4) $|Pi(\boldsymbol{H}(D;a,b))| = 2 \Longrightarrow$ *For any primitive idempotent* $e \in Pi(\boldsymbol{H}(D;a,b))$,

$$\boldsymbol{H}(D;a,b) \cong \begin{pmatrix} e\boldsymbol{H}(D;a,b)e & e\boldsymbol{H}(D;a,b)e \\ e\boldsymbol{H}(D;a,b)e & e\boldsymbol{H}(D;a,b)e \end{pmatrix}.$$

(5) $D = F$ *is a commutative field*

$$\Longrightarrow \quad \boldsymbol{H}(F;a,b) \text{ is a division ring} \quad \text{or} \quad \boldsymbol{H}(F;a,b) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}.$$

(6) *For a commutative field* F, $|Pi(\boldsymbol{H}(F;a,b))| = 4$ *does not occur.*

The following results hold:

Theorem 10. *The following conditions are equivalent:*
(1) $|Pi(\boldsymbol{H}(D;a,b))| = 4$.
(2) *There exist* $p, q, r \in D$ *such that* $p^2 = a, q^2 = b, pq = r = -qp$.

In these case, the following $\{g_i\}_i$ *are orthogonal primitive idempotents:*

$$g_1 = \frac{1}{4}(1 + ipa^{-1} + jqb^{-1} + kr(ab)^{-1}),$$
$$g_2 = \frac{1}{4}(1 + ipa^{-1} - jqb^{-1} - kr(ab)^{-1}),$$
$$g_3 = \frac{1}{4}(1 - ipa^{-1} + jqb^{-1} - kr(ab)^{-1}),$$
$$g_4 = \frac{1}{4}(1 - ipa^{-1} - jqb^{-1} + kr(ab)^{-1}).$$

Theorem 11. *Let* D *be a division ring with* $2 \neq 0$. *Then the following conditions are equivalent.*
(1) $x^2 \neq a$ *for all* $x \in D$.
(2) $\boldsymbol{C}(D;a)$ *is a division ring.*

Proof. (1) $\Rightarrow$ (2). Assume that $\boldsymbol{C}(D;a)$ is not a division ring. Then there exists a primitive idempotent $e = x + iy \neq 1$. Since $\boldsymbol{C}(D;a) = e\boldsymbol{C}(D;a) \oplus (1-e)\boldsymbol{C}(D;a)$, we see that $e\boldsymbol{C}(D;a) = eD$. Hence we can take p in D such that $ei = ep$. Hence it follows $ya + xi = xp + ypi$. Hence $ya = xp$ and $x = yp$ and hence $ya = yp^2$. Therefore $a = p^2$, a contradiction.

(2) $\Rightarrow$ (1). Assume that there exists $x \in D$ such that $x^2 = a$. Put $e = \frac{1}{2}(1 + ix^{-1})$. Then e is an idempotent. Since $\boldsymbol{C}(D;a)$ is a division ring, e must be 1, a contradiction. $\square$

We skip to state the structures of $\boldsymbol{H}(\boldsymbol{H}(D;a,b);c,d)$, $\boldsymbol{H}(\boldsymbol{C}(D;a);c,d)$, $\boldsymbol{C}(\boldsymbol{C}(D;a);b)$ etc.

Finally we shall comment consistency between a classical theorem on $\boldsymbol{H}(F;a,b)$ and our theory on $Pi(F;a,b)$, where F is a field with $2 \neq 0$.

The following is known as a classical theory on $\boldsymbol{H}(F;a,b)$ with $2 \neq 0$.

The following are equivalent:

(1) $\boldsymbol{H}(F;a,b) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.

(2) The equation $X^2 - aY^2 - bZ^2 + abW^2 = 0$ has a non-trivial solution in F.

(3) The equation $X^2 - aY^2 - bZ^2 = 0$ has a non-trivial solution in F.

But the following conditions are **not** equivalent to these conditions.

(4) The equation $X^2 - aY^2 = 0$ has a non-trivial solution in F.

On the other hand, from our theory, we can show that the following are equivalent:

(1) $\boldsymbol{H}(F:a,b) \cong \begin{pmatrix} F & F \\ F & F \end{pmatrix}$.

(2') The equation $\frac{1}{4} - aY^2 - bZ^2 + abW^2 = 0$ has a solution in F.

(3') The equation $\frac{1}{4} - aY^2 - bZ^2 = 0$ has a solution in F, or the equation $\frac{1}{4} + abW^2 = 0$ has a solution in F.

(2") There exists an idempotent e of the form $e = \frac{1}{2} + ix + jy + kz \in \boldsymbol{H}(F;a,b)$.

(3") There exists an idempotent e of the form $e = \frac{1}{2} + ix + jy \in \boldsymbol{H}(F;a,b)$, or e of the form $e = \frac{1}{2} + kw \in \boldsymbol{H}(F;a,b)$.

By way of parenthesis, we shall state (2), (3) imply (2'), (3'), (2"), (3").

Assume the equation $X^2 - aY^2 - bZ^2 + abW^2 = 0$ has a non-trivial solution, say (x,y,z,w). If $x \neq 0$, then $\frac{1}{4} - a(y(2x)^{-1})^2 - b(z(2x)^{-1})^2 + ab(w(2x)^{-1})^2 = 0$, and hence $e = \frac{1}{2} + iy(2x)^{-1} + jz(2x)^{-1} + kw(2x)^{-1}$ is an idempotent.

If $x = 0$, then $-ay^2 - bz^2 + abw^2 = 0$. Here assume $w \neq 0$. Then by calculation, we see $\frac{1}{4} - a(z(2aw)^{-1})^2 - b(y(2bw)^{-1})^2 = 0$, and hence $e = \frac{1}{2} + iz(2aw)^{-1} + jy(2bw)^{-1}$ is an idempotent.

Furthermore, assume that $x = 0$ and $w = 0$. Then $ay^2 + bz^2 = 0$ and it follows $(ay)^2 + abz^2 = 0$, from which we see $e = \frac{1}{2} + kz(2ay)^{-1}$ is an idempotent.

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