ON DELTA INVARIANTS OF CERTAIN IDEALS

TOSHINORI KOBAYASHI

Abstract. We observe the delta invariants of ideals. Auslander and Yoshino gave characterizations of regular local rings in terms of the delta invariants of the maximal ideal. In this paper, we generalize this result to ideals with some extra conditions and give equivalent conditions of ideals to be parameter ideals of local rings. At the end of this paper, we verify that Ulrich ideals satisfy the condition of the main theorem of this paper.

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring with a canonical module. The Auslander δ-invariant $\delta_R(M)$ for a finitely generated R-module M is defined to be the rank of maximal free summand of the minimal Cohen-Macaulay approximation of M. For an integer $n \geq 0$, the n-th δ-invariant is defined by Auslander, Ding and Solberg [2] as $\delta_R^n(M) = \delta_R(\Omega_R^n M)$, where $\Omega_R^n M$ denotes the n-th syzygy module of M in the minimal free resolution.

On these invariants, combining the Auslander's result (see [2, Corollary 5.7]) and Yoshino's one [8], we have the following theorem.

Theorem 1 (Auslander, Yoshino). *Let $d > 0$ be the Krull dimension of R. Consider the following conditions.*

(a) *R is a regular local ring.*
(b) *There exists $n \geq 0$ such that $\delta^n(R/\mathfrak{m}) > 0$.*
(c) *There exist $n > 0$ and $l > 0$ such that $\delta^n(R/\mathfrak{m}^l) > 0$.*

Then, the implications (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) *hold. The implication* (c) $\Rightarrow$ (a) *holds if* $\operatorname{depth} \mathsf{gr}_{\mathfrak{m}}(R) \geq d - 1$.

Here we denote by $\mathsf{gr}_I(R)$ the associated graded ring of R with respect to an ideal I of R. In this paper, we characterize parameter ideals in terms of (higher) δ-invariants.

Theorem 2. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with a canonical module ω, having infinite residue field k and Krull dimension $d > 0$. Let I be an $\mathfrak{m}$-primary ideal of R such that I/I^2 is a free R/I-module. Consider the following conditions.*

(a) *$\delta(R/I) > 0$.*
(b) *I is a parameter ideal of R.*
(c) *There exists $n \geq 0$ such that $\delta^n(R/I) > 0$.*
(d) *There exist $n > 0$ and $l > 0$ such that $\delta^n(R/I^l) > 0$.*

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Then, the implications (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Rightarrow$ (d) *hold. The implication* (d) $\Rightarrow$ (c) *holds if* $\operatorname{depth}\mathsf{gr}_I(R) \geq d-1$ *and* I^i/I^{i+1} *is a free* R/I*-module for any* $i > 0$*. The implication* (b) $\Rightarrow$ (a) *holds if* $I \subset \mathsf{tr}(\omega)$*.*

Here $\mathsf{tr}(\omega)$ is the trace ideal of ω. that is, the image of the natural homomorphism $\omega \otimes_R \mathsf{Hom}_R(\omega, R) \to R$ mapping $x \otimes f$ to $f(x)$ for $x \in \omega$ and $f \in \mathsf{Hom}_R(\omega, R)$. This result recovers Theorem 1 by letting $I = \mathfrak{m}$.

There are some examples of ideals which satisfy the whole conditions in Theorem 2. One of them is the maximal ideal $\mathfrak{m}$ in the case where $\mathsf{gr}_{\mathfrak{m}}(R)$ is Cohen-Macaulay (for example, R is a hypersurface or a localization of a homogeneous graded Cohen-Macaulay ring.) Other interesting examples are Ulrich ideals. These ideals are defined in [3] and many examples of Ulrich ideals are given in [3] and [4]. We shall show that Ulrich ideals satisfy the assumptions of Theorems 2.

2. Sketch of the proof

Throughout this section, let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay local ring of dimension $d > 0$ with a canonical module ω, and assume that k is infinite. We recall some basic properties of the Auslander δ-invariant.

For a finitely generated R-module M, a short exact sequence

$$0 \to Y \to X \xrightarrow{p} M \to 0 \tag{2.1}$$

is called a Cohen-Macaulay approximation of M if X is a maximal Cohen-Macaulay R-module and Y has finite injective dimension over R. We say that the sequence (2.1) is minimal if each endomorphism ϕ of X with $p \circ \phi = p$ is an automorphism of X. It is known (see [1], [5]) that a minimal Cohen-Macaulay approximation of M exists and is unique up to isomorphism.

If the sequence (2.1) is a minimal Cohen-Macaulay approximation of M, then we define the (Auslander) δ-invariant $\delta(M)$ of M as the maximal rank of a free direct summand of X. We denote by $\delta^n(M)$ the δ-invariant of n-th syzygy $\Omega^n M$ of M in the minimal free resolution for $n \geq 0$.

We prepare some basic properties of delta invariants in the next Lemma; see [6, Corollary 11.28].

Lemma 3. *Let M and N be finitely generated R-modules.*

1. *If there exists a surjective homomorphism $M \to N$, then $\delta(M) \geq \delta(N)$.*
2. *The equality $\delta(M \oplus N) = \delta(M) + \delta(N)$ holds true.*

Lemma 4. *Let N be a maximal Cohen-Macaulay R-module. Then $\delta^1(N) = 0$. In particular, $\delta^n(M) = 0$ for $n \geq d+1$ and any finitely generated R-module M.*

We now remark on δ-invariants under reduction by a regular element. The following lemma is shown in [7, Corollary 2.5].

Lemma 5. *Let M be a finitely generated R-module and $x \in \mathfrak{m}$ be a regular element on M and R. If $0 \to Y \to X \to M \to 0$ is a minimal Cohen-Macaulay approximation of M, then*

$$0 \to Y/xY \to X/xX \to M/xM \to 0$$

is a minimal Cohen-Macaulay approximation of M/xM over $R/(x)$. In particular, it holds that $\delta_R(M) \le \delta_{R/(x)}(M/xM)$.

We prepare two more lemmas.

Lemma 6. *Let $l > 0$ be an integer, I be an $\mathfrak{m}$-primary ideal of R and $x \in I \setminus I^2$ be an R-regular element. Assume that I^i/I^{i+1} is a free R/I-module for any $1 \le i \le l$ and the multiplication map $x : I^{i-1}/I^i \to I^i/I^{i+1}$ is injective for any $1 \le i \le l$, where we set $I^0 = R$. Then the following hold.*

(1) *$xI^i = (x) \cap I^{i+1}$ for all $0 \le i \le l$.*
(2) *$I^i/I^{i+1} \cong I^{i-1}/I^i \oplus I^i/(xI^{i-1} + I^{i+1})$ for all $1 \le i \le l$.*
(3) *$I^i/xI^i \cong I^{i-1}/I^i \oplus I^i/xI^{i-1}$ for all $1 \le i \le l$.*

Lemma 7. *Assume $d \le 1$ and I is an $\mathfrak{m}$-primary ideal of R. If $\delta(I) > 0$, then I is a parameter ideal of R.*

Now we can prove Theorem 2.

Proof of Theorem 2. (b) $\Rightarrow$ (c): If I is a parameter ideal, then $\Omega^d(R/I) = R$ and hence $\delta^d(R/I) = 1 > 0$.

(a), (c) $\Rightarrow$ (b): Assume that $\delta(R/I) > 0$. Then the inequality $\delta(I) > 0$ also holds because I/I^2 is a free R/I-module and thus there is a surjective homomorphism $I \to R/I$. Therefore we only need to prove the implication (c) $\Rightarrow$ (b) in the case $n > 0$. We show the implication by induction on the dimension d.

If $d = 1$, then $n = 1$ by Lemma 4. Using Lemma 7, it follows that I is a parameter ideal.

Now let $d > 1$. Take $x \in I \setminus \mathfrak{m}I$ to be an R-regular element. Then the image of x in the free R/I-module I/I^2 forms a part of a free basis over R/I. This provides that the map $x : R/I \to I/I^2$ is injective. We see from Lemma 5 that

$$\begin{aligned} \delta^{n-1}_{R/(x)}(I/xI) &= \delta_{R/(x)}(\Omega^{n-1}_{R/(x)}(I/xI)) \\ &= \delta_{R/(x)}(\Omega^{n-1}_R(I) \otimes_R R/(x)) \\ &\ge \delta_R(\Omega^{n-1}_R I) = \delta^n_R(R/I) > 0. \end{aligned} \tag{2.2}$$

Applying Lemma 6 (3) to $i = 1$, we have an isomorphism $I/xI \cong R/I \oplus I/(x)$ and hence we obtain an equality

$$\delta^{n-1}_{R/(x)}(I/xI) = \delta^{n-1}_{R/(x)}(R/I) + \delta^{n-1}_{R/(x)}(I/(x)).$$

It follows from (2.2) that $\delta^{n-1}_{R/(x)}(R/I) > 0$ or $\delta^{n-1}_{R/(x)}(I/(x)) > 0$. Note that the ideal $\overline{I} := I/(x)$ of $\overline{R} := R/(x)$ satisfies the same condition as (c), that is, the module $\overline{I}/\overline{I}^2$ is free over $\overline{R}/\overline{I} = R/I$, because $\overline{I}/\overline{I}^2 = I/((x)+I^2)$ is a direct summand of I/I^2 by Lemma 6 (2). By the induction hypothesis, the ideal $\overline{I}$ is a parameter ideal of $\overline{R}$. Then we see that I is also a parameter ideal of R.

The implication (c) $\Rightarrow$ (d) is trivial. The implication (d) $\Rightarrow$ (b) follows by induction and similar argument. Finaly, the implication (b) $\Rightarrow$ (a) follows from the proof of [6, Theorem 11.42]. $\square$

Next, we prove that Ulrich ideals satisfies the assumption of Theorem 2. Let I be an $\mathfrak{m}$-primary ideal of R. To begin with, let us recall the definition of Ulrich ideals.

Definition 8. We say that I is an Ulrich ideal of R if it satisfies the following.

(1) $\mathsf{gr}_I(R)$ is a Cohen-Macaulay ring with $a(\mathsf{gr}_I(R)) \leq 1-d$.
(2) I/I^2 is a free R/I-module.

Here we denote by $a(\mathsf{gr}_I(R))$ the a-invariant of $a(\mathsf{gr}_I(R))$. Since k is infinite, the condition (1) of Definition 8 is equivalent to saying that $I^2 = QI$ for some minimal reduction Q of I.

Proposition 9. *Let I be an Ulrich ideal of R. Then I^l/I^{l+1} is free over R/I for any $l \geq 1$.*

Proof. By definition, I/I^2 is free over R/I. Take a minimal reduction Q of I. Consider the canonical exact sequence

$$0 \to I^l/Q^l \to Q^{l-1}/Q^l \to Q^{l-1}/I^l \to 0$$

of R/Q-modules. Then Q^{l-1}/Q^l is a free R/Q-module and

$$Q^{l-1}/I^l = Q^{l-1}/IQ^{l-1} = R/I \otimes_{R/Q} Q^{l-1}/Q^l$$

is a free R/I-module. Therefore

$$I^l/Q^l = \Omega_{R/Q}((R/I)^{\oplus m}) = \Omega_{R/Q}(R/I)^{\oplus m} = (I/Q)^{\oplus m}$$

for some m. Since I/Q is free over R/I, I^l/Q^l is also a free R/I-module. We now look at the canonical exact sequence $0 \to Q^l/I^{l+1} \to I^l/I^{l+1} \to I^l/Q^l \to 0$ of R/I-modules. Then as we already saw, I^l/Q^l and Q^l/I^{l+1} are both free over R/I. Thus the sequence is split exact and I^l/I^{l+1} is a free R/I-module. □

References

[1] M. Auslander and R. O. Buchweitz, The homological theory of maximal CohenMacaulay approximations, Mem. Soc. Math. France 38 (1989), 5-37.

[2] M. Auslander, S. Ding, Ø. Solberg, Liftings and weak liftings of modules, *J. Algebra* 156 (1993), 273–317.

[3] S. Goto, K.Ozeki, R. Takahashi, K. Watanabe, K. Yoshida, Ulrich ideals and modules, *Math. Proc. Cambridge Philos. Soc.* 156 (2014), no. 1, 137–166.

[4] S. Goto, K.Ozeki, R. Takahashi, K. Watanabe, K. Yoshida, Ulrich ideals and modules over two-dimensional rational singularities, *Nagoya Math.* J. 221 (2016), no. 1, 69–110.

[5] M. Hashimoto and A. Shida, Some remarks on index and generalized Loewy length of a Gorenstein local ring, *J. Algebra* 187 (1997), no. 1, 150-162.

[6] G. J. Leuschke; R. Wiegand, Cohen-Macaulay Representations, Mathematical Surveys and Monographs, vol. 181, *American Mathematical Society, Providence, RI*, 2012.

[7] K.Yoshida, A note on minimal Cohen-Macaulay approximations, Communications in Algebra 24(1) (1996), 235–246.

[8] Y. Yoshino, On the higher delta invariants of a Gorenstein local ring, *Proc. Amer. Math. Soc.* 124 (1996), 2641–2647.

Graduate School of Mathiematics
Nagoya University
E-mail address: m16021z@math.nagoya-u.ac.jp