THE ORDINARY QUIVERS OF HOCHSCHILD EXTENSION ALGEBRAS FOR SELF-INJECTIVE NAKAYAMA ALGEBRAS

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Abstract. Let T be a Hochschild extension algebra of a finite dimensional algebra A over an algebraically closed field K by the standard duality A-bimodule $\mathrm{Hom}_K(A, K)$. In this paper, we determine the ordinary quiver of T if A is a self-injective Nakayama algebra by means of the $\mathbb{N}$-graded second Hochschild homology group $HH_2(A)$ in the sense of Sköldberg.

1. Introduction

Throughout the paper, an algebra means a finite dimensional algebra over an algebraically closed field K. Let A be an algebra and D the standard duality functor $\mathrm{Hom}_K(-, K)$. Note that $D(A) = \mathrm{Hom}_K(A, K)$ is an A-bimodule. First, we recall the definition of a Hochschild extension algebra and some basic properties.

Definition 1. An *Hochschild extension over A with kernel* $D(A)$ is an exact sequence

$$0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0$$

such that T is a K-algebra, ρ is an algebra epimorphism and κ is a T-bimodule monomorphism from ${}_\rho(D(A))_\rho$, where ${}_\rho(D(A))_\rho$ is regarded as a T-bimodule by means of ρ. Then T is called a *Hochschild extension algebra of A by* $D(A)$.

Definition 2. An extension $0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0$ is said to be *splittable* if there is an algebra homomorphism $\rho' : A \to T$ with $\rho\rho' = \mathrm{id}_A$.

Definition 3. Two extensions (F) and (F') over A with kernel $D(A)$ are *equivalent* if there exists a K-algebra homomorphism $\iota : T \to T'$ such that the diagram

$$\begin{tikzcd}
(F) & 0 \arrow[r] & D(A) \arrow[r] \arrow[d, "1"] & T \arrow[r] \arrow[d, "\iota"] & A \arrow[r] \arrow[d, "1"] & 0 \\
(F') & 0 \arrow[r] & D(A) \arrow[r] & T' \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

is commutative. The set of all equivalent classes of extensions over A by $D(A)$ is denoted by $F(A, D(A))$.

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A K-bilinear map $\alpha : A \times A \to D(A)$ is said to be 2-cocycle if α satisfies the relation

$$a\alpha(b, c) - \alpha(ab, c) + \alpha(a, bc) - \alpha(a, b)c = 0$$

for $a, b, c \in A$. Using a 2-cocycle α, we define an associative multiplication in the K-vector space $A \oplus D(A)$ by the rule:

$$(a, x)(b, y) = (ab, ay + xb + \alpha(a, b)) \tag{1.1}$$

for $(a, x), (b, y) \in A \oplus D(A)$. Then it is easy to see that $T_\alpha(A) := A \oplus D(A)$ is an associative K-algebra with identity $(1_A, -\alpha(1, 1))$, and that there exists an extension over A by $D(A)$:

$$0 \longrightarrow D(A) \longrightarrow T_\alpha(A) \longrightarrow A \longrightarrow 0.$$

Conversely, given an extension $0 \to D(A) \to T \to A \to 0$, we easily see that T is isomorphic to $T_\alpha(A)$ for some 2-cocycle α.

We identify a 2-cocycle α with the composition of the map $A \times A \to A \otimes A$; $(a, b) \mapsto a \otimes b$ and α. Notice that, by this identification, α is a representative element of a element of Hochschild cohomology $H^2(A, D(A)) := \mathrm{Ext}_{A^e}(A, D(A))$, where $A^e = A \otimes A^{op}$.

Proposition 4 ([2, Proposition 6.2], [6, Section 2.5])**.** *The set $F(A, D(A))$ is in a one-to-one correspondence with $H^2(A, D(A))$. This correspondence $H^2(A, D(A)) \to F(A, D(A))$ is obtained by assigning to each 2-cocycle α, the extension $T_\alpha(A)$. The zero element in $H^2(A, D(A))$ is correspond to the class of splittable extensions.*

The standard duality induces the isomorphism $H^*(A, D(A)) \cong D(HH_*(A))$. So we consider Hochschild homology group $HH_*(A)$ in next section.

2. Hochschild homology groups for truncated quiver algebras

Let Δ be a finite quiver and K a field. We fix a positive integer $n \geq 2$. A *truncated quiver algebra* is defined by $K\Delta/R_\Delta^n$, where R_Δ is the arrow ideal of $K\Delta$ and R_Δ^n is the two-sided ideal of $K\Delta$ generated by the paths of length n. We denote by Δ_0, Δ_1 and Δ_i the set of vertices, the set of arrows and the set of paths of length i, respectively. We put $\Delta_+ = \bigcup_{i=1}^\infty \Delta_i$. For a truncated quiver algebra, Sköldberg given a projective resolution and computed Hochschild homology groups.

Theorem 5 (See [5, Theorem 1])**.** *Let A be a truncated quiver algebra $K\Delta/R_\Delta^n$. Then we have the following projective resolution of A as a left A^e-module:*

$$\begin{aligned}\boldsymbol{P} : \cdots \longrightarrow A \otimes_{K\Delta_0} K\Delta_{n+1} \otimes_{K\Delta_0} A \xrightarrow{d_3} A \otimes_{K\Delta_0} K\Delta_n \otimes_{K\Delta_0} A \\ \xrightarrow{d_2} A \otimes_{K\Delta_0} K\Delta_1 \otimes_{K\Delta_0} A \xrightarrow{d_1} A \otimes_{K\Delta_0} A \xrightarrow{d_0} A \longrightarrow 0.\end{aligned}$$

Here the differentials d_2 and d_3 are defined by

$$d_2(x \otimes y_1 \cdots y_n \otimes z) = \sum_{j=0}^{n-1} x \otimes y_1 \cdots y_j \otimes y_{j+1} \otimes y_{j+2} \cdots y_n z$$

and

$$d_3(x \otimes y_1 \cdots y_{n+1} \otimes z) = xy_1 \otimes y_2 \cdots y_{n+1} \otimes z - x \otimes y_1 \cdots y_n \otimes y_{n+1} z,$$

for $x, z \in A$ and $y_i \in \Delta_1$ $(1 \leq i \leq n+1)$.

We denote by $\boldsymbol{P}_i$ the i th term of $\boldsymbol{P}$, then $A \otimes_{A^e} \boldsymbol{P}_i$ is the i th term of $A \otimes_{A^e} \boldsymbol{P}$. We have

$$\begin{aligned} A \otimes_{A^e} \boldsymbol{P}_1 &= A \otimes_{A^e} (A \otimes_{K\Delta_0} K\Delta_1 \otimes_{K\Delta_0} A) \\ &\xrightarrow{\sim} A \otimes_{A^e} A^e \otimes_{K\Delta_0^e} K\Delta_1 \xrightarrow{\sim} A \otimes_{K\Delta_0^e} K\Delta_1. \end{aligned}$$

The set of cycles of length q is denoted by $\Delta_q^{\mathrm{c}}(\subset \Delta_q)$. A cycle γ is a *basic* cycle provided that we can not write $\gamma = \beta^i$, for $i \geq 2$. The set of basic cycles of length q is denoted by Δ_q^{b}. Let C_q be the cyclic group of order q, with generator c. Then we define an action of C_q on Δ_q^{c} by $c(a_1 \cdots a_q) = a_q a_1 \cdots a_{q-1}$. For each $\gamma \in \Delta_q^{\mathrm{c}}$, we define the *orbit* of γ to be the subset $\overline{\gamma} = \{c^i(\gamma) \,|\, 1 \leq i \leq q\} \in \Delta_q^{\mathrm{c}}$. We denote the set of orbits by $\Delta_q^{\mathrm{c}}/C_q$.

Theorem 6 (See [5, Theorem 2])**.** *Let A be a truncated quiver algebra $K\Delta/R_\Delta^n$. Then the degree q part of the second Hochschild homology $HH_{2,q}(A)$ is given by*

$$\begin{aligned} &HH_{2,q}(A) \\ &\quad = \begin{cases} K^{a_q} & \text{if } n+1 \leq q \leq 2n-1, \\ \bigoplus_{r|q} (K^{\gcd(n,r)-1} \oplus \operatorname{Ker}(\cdot \frac{n}{\gcd(n,r)} : K \to K))^{b_r} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Here we set $a_q := \operatorname{card}(\Delta_q^{\mathrm{c}}/C_q)$ and $b_r := \operatorname{card}(\Delta_r^{\mathrm{b}}/C_r)$.

For an dual basis element $u^* := (a_{n+1} \cdots a_r \otimes_{K\Delta_0^e} a_1 a_2 \cdots a_n)^* \in D(A \otimes_{K\Delta_0^e} K\Delta_n)$ $(a_i \in \Delta_1)$, $\Theta(u^*) \in \operatorname{Hom}_K(A^{\otimes 2}, D(A))$ is the map as follows:

$$\begin{aligned} &b_1 \cdots b_{m_1} \otimes_K b_{m_1+1} \cdots b_{m_1+m_2} \\ &\qquad \mapsto \begin{cases} (a_{m_1+m_2+1} \cdots a_r)^* & \text{if } n \leq m_1 + m_2 \leq r \\ & \text{and } b_t = a_t \text{ for } t (1 \leq t \leq m_1+m_2), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The isomorphism $D(HH_2(A)) \cong H^2(A,\, D(A))$ is induced by Θ. Moreover, we get 2-cocycles from $D(HH_{2,q}(A))$ through the following isomorphism:

$$\bigoplus_q D(HH_{2,q}(A)) \cong D(\bigoplus_q HH_{2,q}(A)) = D(HH_2(A)) \xrightarrow{\sim} H^2(A,\, D(A)).$$

We denote the composition of the above isomorphisms by Θ again.

3. The ordinary quivers of Hochschild extension algebras

Lemma 7. *Let Δ be a finite quiver and $A = K\Delta/I$ for an admissible ideal I. Let $T_\alpha(A)$ be an extension algebra of A defined by a 2-cocycle $\alpha : A \times A \to D(A)$. We denote by $\Delta_{T_\alpha(A)}$ and $\Delta_{T_0(A)}$ the ordinary quiver of $T_\alpha(A)$ and the trivial extension algebra $T_0(A)$, respectively. If $\alpha(e_i,\, -) = \alpha(-,\, e_i) = 0$ for all $i \in \Delta_0$, then we have the chain of subquivers of $\Delta_{T_0}(A)$:*

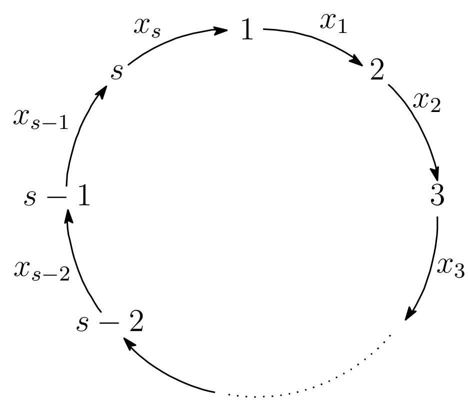
$$\Delta \subseteq \Delta_{T_\alpha(A)} \subseteq \Delta_{T_0(A)}.$$

Lemma 8. *Let Δ be a finite quiver and $A = K\Delta/I$ for an admissible ideal I. Let $T_\alpha(A)$ be an extension algebra of A defined by a 2-cocycle $\alpha : A \times A \to D(A)$. If $\alpha(e_i, -) = \alpha(-, e_i) = 0$ for all $i \in \Delta_0$, then the following conditions are equivalent*:

(1) $\alpha(J(A), J(A)) \subseteq J(A)D(A) + D(A)J(A)$.

(2) $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$.

From now on, we consider self-injective Nakayama algebras. Let Δ be the following cyclic quiver with $s\,(\geq 1)$ vertices and s arrows:



Suppose $n \geq 2$ and $A = K\Delta/R^n_\Delta$, which is called a *truncated cycle algebra* in [1], where R^n_Δ is the two-sided ideal of $K\Delta$ generated by the paths of length n. We regard the subscripts i of e_i and x_i modulo $s\,(1 \leq i \leq s)$. By Theorem 6, the second Hochschild homology is given by

$$HH_{2,\,q}(A) = \begin{cases} K & \text{if } s|q \text{ and } n+1 \leq q \leq 2n-1, \\ K^{s-1} \oplus \mathrm{Ker}(\cdot\frac{n}{s} : K \to K) & \text{if } s|q \text{ and } q = n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

We have the following main theorem about the ordinary quiver of Hochschild extension algebras.

Theorem 9 ([3]). *Suppose that $n \geq 2$, $A = K\Delta/R^n_\Delta$ and $n \leq q \leq 2n-1$. Let $\alpha : A \times A \to D(A)$ be a 2-cocycle such that the cohomology class $[\alpha]$ of α belongs to $\Theta(D(HH_{2,\,q}(A)))$, and let $T_\alpha(A)$ be the Hochschild extension algebra of A defined by α. Then the ordinary quiver $\Delta_{T_\alpha(A)}$ is given by*

$$\Delta_{T_\alpha(A)} = \begin{cases} \Delta_{T_0(A)} & \textit{if} \quad n \leq q \leq 2n-2, \\ \Delta & \textit{if} \quad q = 2n-1. \end{cases}$$

Corollary 10. *Suppose that $n \geq 2$ and $A = K\Delta/R^n_\Delta$. Let $\alpha : A \times A \to D(A)$ be a 2-cocycle and $[\alpha] = \sum_{q=n}^{2n-1}[\beta_q]$, where $\beta_q : A \times A \to D(A)$ is a 2-cocycle such that the cohomology class $[\beta_q]$ of β_q belongs to $\Theta(D(HH_{2,\,q}(A)))$. Then the following equation holds*:

$$\Delta_{T_\alpha(A)} = \begin{cases} \Delta_{T_0(A)} & \textit{if } [\beta_{2n-1}] = 0, \\ \Delta & \textit{if } [\beta_{2n-1}] \neq 0. \end{cases}$$

Corollary 11. *Suppose that $n \geq 2$ and $A = K\Delta/R_\Delta^n$. Let $\alpha : A \times A \to D(A)$ be a 2-cocycle. If $\Delta_{T_\alpha(A)} = \Delta$, then $T_\alpha(A)$ is isomorphic to $K\Delta/R_\Delta^{2n}$ and $T_\alpha(A)$ is symmetric.*

4. Relations for a Hochschild extension algebra

We will investigate the relations dividing into the following two cases: Case 1: $n+1 \leq s \leq 2n-2$ or $(2n-1)/3 < s < n-1/2$, and Case 2: $s = n$.

4.1. **Case 1: $n+1 \leq s \leq 2n-2$ or $(2n-1)/3 < s < n-1/2$.** Let

$$q := \begin{cases} s & \text{if } n+1 \leq s \leq 2n-2, \\ 2s & \text{if } (2n-1)/3 < s < n-1/2. \end{cases}$$

In this case, note that $\dim_K HH_{2,q}(A) = 1$ by (3.1). So we have the following proposition.

Proposition 12. *We define maps $\alpha_i : A \times A \to D(A)$ $(i = 1, \ldots, s)$ by*

$$\alpha_i(a_1 \cdots a_{m_1},\, a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} (\overline{x_{i+m_1+m_2} \cdots x_{i+q-1}})^* & \text{if } n \leq m_1+m_2 \leq q \\ & \quad \text{and } a_t = \overline{x_{i+t-1}} \text{ for } 1 \leq t \leq m_1+m_2, \\ 0 & \text{otherwise,} \end{cases}$$

for an arrow $a_t (1 \leq t \leq m_1+m_2)$, and $\alpha_i(\overline{e},\, -) = \alpha_i(-,\, \overline{e}) = 0$ for a trivial path e. Then $\sum_{i=1}^{s} \alpha_i$ is a 2-cocycle, and the cohomology class $[\sum_{i=1}^{s} \alpha_i]$ is a K-basis of $H^2(A,\, D(A))$.

Theorem 13 ([4])**.** *Let $\alpha = k\sum_{i=1}^{s} \alpha_i$ for $k \in K$, where α_i's are the maps in* Proposition 12. *Let I' be the ideal in $K\Delta_T$ generated by*

$$x_i y_{i+1} - y_i x_{i-n+1}, \quad y_i y_{i-n+1},$$
$$x_i x_{i+1} \cdots x_{i+n-1} - k y_i x_{i-n+1} x_{i-n+2} \cdots x_{i-n+(2n-q-1)}$$

for $i = 1, 2, \ldots, s$. Then I' is admissible and $I' = I_{T_\alpha(A)}$. So $T_\alpha(A)$ is isomorphic to $K\Delta_T/I'$.

4.2. **Case 2: $s = n$.** In this case, we note that $\dim_K HH_{2,s}(A) = s-1$ by (3.1). We have the following proposition.

Proposition 14. *We define maps $\alpha_i : A \times A \to D(A)$ $(i = 1, 2, \ldots, s-1)$ by*

$$\alpha_i(a_1 \cdots a_{m_1},\, a_{m_1+1} \cdots a_{m_1+m_2}) = \begin{cases} \overline{e_i}^* & \text{if } m_1+m_2 = s \\ & \quad \text{and } a_t = \overline{x_{i+t-1}} \text{ for } 1 \leq t \leq s, \\ 0 & \text{otherwise,} \end{cases}$$

for an arrow $a_t (1 \leq t \leq m_1+m_2)$, and $\alpha_i(\overline{e},\, -) = \alpha_i(-,\, \overline{e}) = 0$ for a trivial path e. Then α_i's are 2-cocycles, and the set of the cohomology classes is a K-basis of $H^2(A,\, D(A))$.

Theorem 15 ([4])**.** *Let $\alpha = \sum_{i=1}^{s-1} k_i \alpha_i$ for $k_i \in K$, where α_i's are the 2-cocycles as in* Proposition 14. *Let I' be the ideal in $K\Delta_{T_\alpha(A)}$ generated by*

$$x_j y_{j+1} - y_j x_{j+1}, \quad y_j y_{j+1}, \quad x_s x_1 \cdots x_{s-1},$$
$$x_l x_{l+1} \cdots x_{l+s-1} - k_l y_l x_{l+1} \cdots x_{l+s-1}$$

for $j = 1, 2, \ldots, s$ *and* $l = 1, 2, \ldots, s-1$*. Then* I' *is admissible and* $I' = I_{T_\alpha(A)}$*. So* $T_\alpha(A)$ *is isomorphic to* $K\Delta_T/I'$*.*

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