# TWO-SIDED TILTING COMPLEXES AND FOLDED TREE-TO-STAR COMPLEXES

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Abstract. In this note, we explain how to construct two-sided tilting complexes corresponding to Rickard tree-to-star complexes, and give operations for the two-sided tilting complexes corresponding to foldings which are operations for tree-to-star complexes.

## 1. Introduction

For finite dimensional symmetric algebras $\Gamma$ and $\Lambda$ over an algebraically closed field $k$, the following is known.

**Theorem 1.** [6, 7] *Let $\Gamma$ and $\Lambda$ be symmetric algebras. Then the following are equivalent.*
(1) *$\Gamma$ and $\Lambda$ are derived equivalent.*
(2) *There exists a complex $T$ of $K^b(\Gamma\text{-proj})$ which satisfies the following conditions.*
  (i) $\mathrm{Hom}_{K^b(\Gamma\text{-proj})}(T, T[n]) = 0$ $(0 \neq \forall n \in \mathbb{Z})$.
  (ii) $\mathrm{add}(T)$ *generates* $K^b(\Gamma\text{-proj})$ *as a triangulated category.*
  (iii) $\mathrm{End}_{K^b(\Gamma\text{-proj})}(T) \cong \Lambda^{op}$.
(3) *There exist a complex $C$ in $D^b(\Gamma \otimes_k \Lambda^{op})$ satisfying the following condtions for some complex $D$ in $D^b(\Lambda \otimes_k \Gamma^{op})$:*

$$C \otimes^{\mathbb{L}}_{\Lambda} D \cong \Gamma \text{ in } D^b(\Gamma \otimes_k \Gamma^{op}) \text{ and } D \otimes^{\mathbb{L}}_{\Gamma} C \cong \Lambda \text{ in } D^b(\Lambda \otimes_k \Lambda^{op})$$

**Definition 2.** A complex $T$ over $\Gamma$ is called a one-sided tilting complex if it satisfies the conditions (i) and (ii) in Theorem 1 (2). A complex $C$ over $\Gamma \otimes_k \Lambda^{op}$ is called a two-sided tilting complex if it satisfies the conditions in Theorem 1 (3).

For a one-sided tilting complex $T$ over $\Gamma$ with endomorphism algebra $\Lambda^{op}$, it is known that there exists a two-sided tilting complex $D_T$ over $\Gamma \otimes_k \Lambda^{op}$ such that $D_T \cong T$ in $D^b(\Gamma)$ by [7] and [2]. In general, it is hard to describe a derived equivalence induced by the one-sided tilting complex $T$ concretely by using the description of $T$. On the other hand, for the two-sided tilting complex $D_T$, a derived equivalence between $D^b(\Lambda)$ and $D^b(\Gamma)$ is given by $D_T \otimes^{\mathbb{L}}_{\Lambda} - : D^b(\Lambda) \to D^b(\Gamma)$. Moreover, if each term of $D_T$ is projective as a $\Gamma$-module and as a $\Lambda^{op}$-module, then $D_T \otimes^{\mathbb{L}}_{\Lambda} - \cong D_T \otimes_{\Lambda} -$. This derived equivalence can be described concretely and directly by using the description of $D_T$. In this sense, for the one-sided tilting complex $T$, if we construct such a two-sided tilting complex $D_T$ by using concrete bimodules projective on both sides, we get an explicit derived equivalence between $D^b(\Gamma)$ and $D^b(\Lambda)$ induced by the one-sided tilting complex $T$. We aim at constructing explicit two-sided tilting complexes isomorphic to Rickard tree-to-star complexes constructed in [6, Section 4] which are one-sided tilting complexes

The detailed versions [4] and [3] of this article will be published.

over Brauer tree algebras whose endomorphism algebras are isomorphic to Brauer star algebras. Moreover we give operations for the two-sided tilting complexes corresponding to foldings introduced in [8] which are operations for Rickard tree-to-star complexes.

Throughout the rest of this paper, $k$ means an algebraically closed field, $A$ means a Brauer tree algebra over $k$ associated to a Brauer tree with $e$ edges and multiplicity $\mu$ of the exceptional vertex and $B$ means a Brauer tree algebra over $k$ with respect to a "star" with $e$ edges and exceptional vertex with multiplicity $\mu$ in the center (or equivalently is a symmetric Nakayama algebra over $k$ with $e$ simple modules and the nilpotency degree of the radical being $e\mu + 1$). For an edge corresponding to a simple $A$-module $L$, we define a positive integer $d(L)$ as the distance from the exceptional vertex to the furthest vertex of the edge. On this definition we put $m := \max\{d(L) \mid L : \text{simple } A\text{-module}\}$. Moreover let $S$ be a simple $A$-module such that $d(S) = m$.

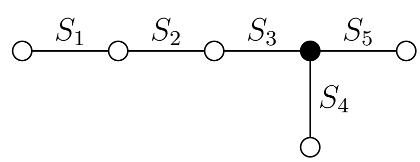
## 2. RICKARD'S RESULT AND RICKARD-SCHAPS'S RESULT

2.1. **RICKARD'S RESULT.** In [6], Rickard classified the derived equivalence classes of Brauer tree algebras.

**Theorem 3.** [6] *Two Brauer tree algebras are derived equivalent if and only if the Brauer trees have the same number of edges and the same multiplicity.*

For the proof of this fact, Rickard constructed a one-sided tilting complex $T$ over $A$ with $\mathrm{End}_{K^b(A\text{-proj})}(T) \cong B^{op}$. We call this one-sided tilting complex $T$ Rickard tree-to-star complex.

**Example 4.** Let $A$ be a Brauer tree algebra associated to the following Brauer tree.




Then Rickard tree-to-star complex $T$ is given by as follows:

$$3P(S_3) \oplus P(S_4) \oplus P(S_5) \to 2P(S_2) \to P(S_1)$$

2.2. **RICKARD-SCHAPS'S RESULT.** In [8], Rickard and Schaps gave operations on the Rickard tree-to-star complexes, called foldings, which produce other tree-to-star complexes. The tree-to-star complexes constructed in [8] are called Rickard-Schaps tree-to-star complexes.

**Definition 5.** Let $T$ be a tree-to-star complex. The following two kinds of operations on $T$ are called foldings.

(1) $-2$ shift of $P(i)$ in the leftmost nonzero term of $T$ where the edge $i$ is not adjacent to the exceptional vertex.
(2) $-2$ shift of $\bigoplus P(i)$ in the leftmost nonzero term of $T$ where the edge $i$ runs over all the edges adjacent to the exceptional vertex.

**Theorem 6.** [8] *Let $T$ be a Rickard tree-to-star complex, and $T'$ a complex obtained by applying foldings to $T$ any times. Then $T'$ is a tree-to-star complex again. In other word, for Rickard tree-to-star complex $T$, each $T_i$ in below diagram is a tree-to-star complex:*

$$T \xrightarrow{\text{folding}} T_1 \xrightarrow{\text{folding}} T_2 \longrightarrow \cdots \longrightarrow T_{n-1} \xrightarrow{\text{folding}} T_n = T'.$$

**Example 7.** Let $A$ be a Brauer tree algebra in Example 4, and $T$ a Rickard tree-to-star complex in Example 4. Then the following complex given by applying folding to $T$ is tree-to-star complex again.

$$2P(S_2) \to P(S_1) \oplus 3P(S_3) \oplus P(S_4) \oplus P(S_5)$$

We denote this tree-to-star complex by $T_1$. The following complex given by applying folding to $T_1$ is also tree-to-star complex.

$$P(S_1) \oplus 3P(S_3) \oplus P(S_4) \oplus P(S_5) \to 2P(S_2)$$

We denote this tree-to-star complex by $T_2$.

## 3. TWO-SIDED TILTING COMPLEXES CORRESPONDING TO TREE-TO-STAR COMPLEXES

In this section, we construct two-sided tilting complexes corresponding to Rickard tree-to-star complexes and Rickard-Schaps tree-to-star complexes.

### 3.1. **PRELIMINARIES.**

**Definition 8.** [1] Let $\Gamma$ and $\Lambda$ be symmetric algebras. Then $\Gamma$ and $\Lambda$ are said to be stably equivalent of Morita type if there exists a $\Gamma \otimes_k \Lambda^{op}$-module $M$ such that

(1) $M$ is projective as a $\Gamma$-module and as a $\Lambda^{op}$-module,
(2) $M \otimes_\Lambda M^* \cong \Gamma \oplus P$ as $\Gamma \otimes_k \Gamma^{op}$-modules, where $P$ is a finitely generated projective $\Gamma \otimes_k \Gamma^{op}$-module and where $M^* = \mathrm{Hom}_k(M, k)$.

**Proposition 9.** [7] *Two derived equivalent symmetric algebras are stably equivalent of Morita type.*

We construct two-sided tilting complexes from minimal projective resolutions of bimodules which induce stable equivalences of Morita type. To calculate the minimal projective resolutions, we use Rouquier's result.

**Lemma 10.** [9] *Let $\Gamma$ and $\Lambda$ be symmetric algebras, and let $N$ be a $\Gamma \otimes_k \Lambda^{op}$-module which is projective as a $\Gamma$-module and as a $\Lambda^{op}$-module. Then the projective cover of $N$ is given by*

$$\bigoplus_W P(N \otimes_\Lambda W) \otimes_k P(W)^*$$

*where $W$ runs over a complete set of representatives of isomorphism classes of simple $\Lambda$-modules and where $P(W)^* = \mathrm{Hom}_k(P(W), k)$.*

3.2. **TWO-SIDED TILTING COMPLEXES CORRESPONDING TO RICKARD TREE-TO-STAR COMPLEXES.** In this section, we construct a two-sided tilting complex over $A \otimes_k B^{op}$ isomorphic to the Rickard tree-to-star complex $T$ in $D^b(A)$.

We know the Brauer tree algebras $A$ and $B$ are derived equivalent. Hence they are stably equivalent of Morita type by Proposition 9 since Brauer tree algebras are symmetric algebras. Therefore there exists an indecomposable $A \otimes_k B^{op}$-module $M$ inducing a stable equivalence of Morita type between $A$ and $B$ induced by $T$.

We need the next lemma later.

**Lemma 11.** *For the indecomposable $A \otimes_k B^{op}$-module $M$ and the simple $A$-module $S$ defined in Section 3, $M^* \otimes_A S$ is a simple $B$-module.*

We denote the simple $B$-module $M^* \otimes_A S$ by $V$. Since $B$ is a symmetric Nakayama algebra with $e$ simple modules, $\{\Omega^{2i}V | 0 \leq i \leq e-1\}$ is a complete set of representatives of isomorphism classes of simple $B$-modules. Hence a projective cover of $M$ as an $A \otimes_k B^{op}$-module is given by

$$\bigoplus_{0 \leq i \leq e-1} P(M \otimes_B \Omega^{2i}V) \otimes_k P(\Omega^{2i}V)^*.$$

Also by [5, Theorem 2.1 (ii)], we have

$$M \otimes_B \Omega^{2i}V \cong \Omega^{2i}(M \otimes_B V) \cong \Omega^{2i}S.$$

If $M$ induces stable equivalence of Morita type between $A$ and $B$, then so does $\Omega^n M$ for any integer $n$ by [10, 2.3.5]. Hence by Lemma 10 we obtain a minimal projective resolution of $M$:

$$\begin{aligned}
\cdots \to & \bigoplus_{0 \leq i \leq e-1} P(\Omega^{2i+n-1}S) \otimes_k P(\Omega^{2i}V)^* \\
\to & \cdots \\
\to & \bigoplus_{0 \leq i \leq e-1} P(\Omega^{2i+1}S) \otimes_k P(\Omega^{2i}V)^* \\
\xrightarrow{\pi_1} & \bigoplus_{0 \leq i \leq e-1} P(\Omega^{2i}S) \otimes_k P(\Omega^{2i}V)^* \\
\xrightarrow{\pi_0} & M
\end{aligned}$$

where $\bigoplus_{0 \leq i \leq e-1} P(\Omega^{2i+n-1}S) \otimes_k P(\Omega^{2i}V)^*$ is in the degree $n$.

**Lemma 12.** *For the above projective resolution of $M$ and $1 \leq l \leq m-2$,*

$$\pi_l\Big(\bigoplus_{d(\mathrm{top}(\Omega^{l+2i}S)) \leq m-l-1} P(\Omega^{l+2i}S) \otimes_k P(\Omega^{2i}V)^*\Big)$$

*is contained in*

$$\bigoplus_{d(\mathrm{top}(\Omega^{l-1+2i}S)) \leq m-l} P(\Omega^{l-1+2i}S) \otimes_k P(\Omega^{2i}V)^*.$$

We can construct a two-sided complex $C = (C_n, d_n)$ by deleting a direct summand in each term of the projective resolution of $M$ as follows:

$$\begin{cases} C_0 = M & \\ C_n = \displaystyle\bigoplus_{d(\mathrm{top}(\Omega^{n-1+2i}S)) \le m-n} P(\Omega^{n-1+2i}S) \otimes_k P(\Omega^{2i}V)^* & (1 \le n \le m-1) \\ C_n = 0 & (\text{otherwise}) \end{cases}$$

and letting $d_n$ be the restriction of $\pi_n$ to $C_n$. By Lemma 12 we have that $d_n$ is well-defined for each $n$. This complex $C$ is a two-sided tilting complex and $C \cong T$ in $D^b(A)$.

3.3. **TWO-SIDED TILTING COMPLEXES CORRESPONDING TO RICKARD-SCHAPS TREE-TO-STAR COMPLEXES.** In this section, we realize foldings as operations for two-sided tilting complexes, that is we give operations *two-sided foldings* for two-sided tilting complexes which make the following diagram commutative where vertical arrows mean the restrictions to $A$ from $A \otimes_k B^{op}$ and where $D_{T_i} \cong T_i$ in $D^b(A)$ for each $1 \le i \le n$.

$$\begin{tikzcd} C \arrow[r, "\text{two-sided folding}"] \arrow[d] & D_{T_1} \arrow[r, "\text{two-sided folding}"] \arrow[d] & D_{T_2} \arrow[r] \arrow[d] & \cdots \arrow[r] & D_{T_{n-1}} \arrow[r, "\text{two-sided folding}"] \arrow[d] & D_{T_n} \arrow[d] \\ T \arrow[r, "\text{folding}"] & T_1 \arrow[r, "\text{folding}"] & T_2 \arrow[r] & \cdots \arrow[r] & T_{n-1} \arrow[r, "\text{folding}"] & T_n \end{tikzcd}$$

We define the following operations for two-sided tilting complexes.

**Definition 13.** (1) Deleting the direct summand of the form $P(U) \otimes_k X$ from the leftmost non-zero term and the one of the form $P(U) \otimes_k X'$ from the second leftmost term for a simple module $U$ not adjacent to the exceptional vertex.

(2) Deleting the direct summand of the form $\bigoplus_{i=1}^n P(U_i) \otimes_k X_i$ from the leftmost non-zero term and the one of the form $\bigoplus_{i=1}^n P(U_i) \otimes_k X_i'$ from the second leftmost term for the all simple modules $U_1, \cdots, U_n$ adjacent to the exceptional vertex.

These operations correspond to foldings.

**Theorem 14.** *The operation for tree-to-star complexes in Definition 5 (1) and (2) corresponds to the operation for two-sided tilting complexes in Definition 13 (1) and (2) respectively.*

We remark that in spite of boundedness of the two-sided tilting complex, the two kinds of operations in Definition 13 can be applied to the two-sided tilting complex for any time (for example see Example 16).

## 4. EXAMPLES

In this section, let $A$ be a Brauer tree algebra in Example 4, and $B$ a Brauer star algebra with 5 simple modules and the same multiplicity as the Brauer tree of $A$, that is the derived equivalent Brauer star algebra to $A$. We denote the Rickard tree-to-star complex over $A$ by $T$ (see Example 4), and the Rickard-Schaps tree-to-star complexes in

Example 7 by $T_1$ and $T_2$ respectively. Moreover let $M$ be an indecomposable $A \otimes_k B^{op}$-module inducing a stable equivalence of Morita type induced by $T$, and denote the simple $B$-modules $\mathrm{top}\,(M^* \otimes_A S_i)$ by $V_i$. With these notation, we construct two-sided tilting complexes $C, D_{T_1}$ and $D_{T_2}$ corresponding to $T, T_1$ and $T_2$ respectively.

**Example 15.** We have a minimal projective resolution of $M$ as an $A \otimes_k B^{op}$-module as follows.

$$
\begin{array}{ccccccc}
 & & P(S_2) \otimes P(V_1)^* & & P(S_1) \otimes P(V_1)^* & & \\
 & & \oplus & & \oplus & & \\
 & & P(S_1) \otimes P(V_2)^* & & P(S_2) \otimes P(V_2)^* & & \\
 & & \oplus & & \oplus & & \\
\cdots & \rightarrow & P(S_4) \otimes P(V_3)^* & \rightarrow & P(S_3) \otimes P(V_3)^* & \rightarrow & {}_A M_B \\
 & & \oplus & & \oplus & & \\
 & & P(S_5) \otimes P(V_4)^* & & P(S_4) \otimes P(V_4)^* & & \\
 & & \oplus & & \oplus & & \\
 & & P(S_3) \otimes P(V_5)^* & & P(S_5) \otimes P(V_5)^* & &
\end{array}
$$

By deleting some summand from each term, we have a two-sided tilting complex $C$ of $A \otimes_k B^{op}$-modules isomorphic to the Rickard tree-to-star complex $T$ in $D^b(A)$ as follows.

$$
\begin{array}{ccccc}
 & & P(S_2) \otimes P(V_2)^* & & \\
 & & \oplus & & \\
P(S_4) \otimes P(V_3)^* & \rightarrow & P(S_3) \otimes P(V_3)^* & \rightarrow & {}_A M_B \\
\oplus & & \oplus & & \\
P(S_5) \otimes P(V_4)^* & & P(S_4) \otimes P(V_4)^* & & \\
\oplus & & \oplus & & \\
P(S_3) \otimes P(V_5)^* & & P(S_5) \otimes P(V_5)^* & &
\end{array}
$$

**Example 16.** For the two-sided tilting complex $C$ in Example 15, applying the operation of Definition 13 (2), we have the following two-sided tilting complex $D_{T_1}$ over $A \otimes_k B^{op}$ isomorphic to $T_1$ in $D^b(A)$.

$$
D_{T_1}: \quad P(S_2) \otimes P(V_2)^* \quad \rightarrow \quad {}_A M_B
$$

This two-sided tilting complex coincides with the one constructed by Rouquier in [9]. This complex is isomorphic to the following complex in $D^b(A \otimes_k B^{op})$, where the middle term is the injective hull of $M$.

$$\begin{array}{ccccc}
 & & P(S_1)\otimes P(V_1)^* & & \\
 & & \oplus & & \\
P(S_2)\otimes P(V_2)^* & & P(S_3)\otimes P(V_2)^* & & \\
 & & \oplus & & \\
 & \to & P(S_2)\otimes P(V_3)^* & \to & {}_A\Omega^{-1}M_B \\
 & & \oplus & & \\
 & & P(S_4)\otimes P(V_4)^* & & \\
 & & \oplus & & \\
 & & P(S_5)\otimes P(V_5)^* & &
\end{array}$$

Applying the operation of Definition 13 (1) to this two-sided tilting complex $D_{T_1}$, we have the following two-sided tilting complex $D_{T_2}$ over $A \otimes_k B^{op}$ isomorphic to $T_2$ in $D^b(A)$.

$$\begin{array}{cccc}
 & P(S_1)\otimes P(V_1)^* & & \\
 & \oplus & & \\
D_{T_2}: & P(S_3)\otimes P(V_2)^* & \to & {}_A\Omega^{-1}M_B \\
 & \oplus & & \\
 & P(S_4)\otimes P(V_4)^* & & \\
 & \oplus & & \\
 & P(S_5)\otimes P(V_5)^* & &
\end{array}$$

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