A NECESSARY CONDITION FOR TWO COMMUTATIVE NOETHERIAN RINGS TO BE SINGULARLY EQUIVALENT

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Abstract. In this article, we consider the question that whether singularly equivalent commutative Noetherian rings have homeomorphic singular loci. This question is quite natural indeed this is true for all known examples of singular equivalences. The aim of this article is to prove that the question is true for a certain class of commutative Noetherian rings.

1. Introduction

This article is based on the paper [10]. Let R be a (not necessarily commutative) Noetherian ring. In the 1980s, Buchweitz [7] defined the *stable derived category* of R, which recently called the *singularity category* of R. It is by definition the Verdier quotient

$$\mathsf{D}_{\mathsf{sg}}(R) := \mathsf{D}^{\mathsf{b}}(\mathsf{mod}R)/\mathsf{K}^{\mathsf{b}}(\mathsf{proj}R),$$

where $\mathsf{D}^{\mathsf{b}}(\mathsf{mod}R)$ stands for the bounded derived category of finitely generated R-modules, and $\mathsf{K}^{\mathsf{b}}(\mathsf{proj}R)$ the bounded homotopy category of finitely generated projective R-modules. Singularity categories have been deeply investigated from algebro-geometric and representation-theoretic motivations [8, 9, 14, 15] and connected to Kontsevich's Homological Mirror Symmetry Conjecture by Orlov [12].

For two Noetherian rings R and S, we say that they are *singularly equivalent* if their singularity categories $\mathsf{D}_{\mathsf{sg}}(R)$ and $\mathsf{D}_{\mathsf{sg}}(R)$ are equivalent as triangulated categories. It is well known that Morita equivalences and derived equivalences imply singularly equivalences. Complete characterization for Morita equivalences and derived equivalences are known [11, 13], however singularly equivalences are quite difficult. Indeed, only a few examples of such equivalences are known:

Example 1. (1) $R \cong S \Rightarrow R \overset{\mathsf{sg}}{\approx} S$

(2) R, S: regular $\Rightarrow R \overset{\mathsf{sg}}{\approx} S$

(3) (Knörrer's periodicity, see [17]) Let k be a field of characteristic 0 and $0 \neq f \in (x_0, \dots, x_d)^2 \subseteq k[[x_0, \dots, x_d]]$. Set $R := k[[x_0, x_1, \dots, x_d]]/(f)$ and $S := k[[x_0, x_1, \dots, x_d, u, v]]/(f + uv)$. Then R and S are singularly equivalent.

By simple argument, we can check that singular loci of R and S are homeomorphic. Here, for the third case, we use the Jacobian criterion. Therefore, it is natural to ask that whether singularly equivalent commutative Noetherian rings have homeomorphic singular loci. The aim of this article is to show that this is true for a certain class of commutative Noetherian rings:

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Theorem 2. *Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ be complete intersections which are locally hypersurfaces on the punctured spectra. If R and S are singularly equivalent, then $\mathsf{Sing}\, R$ and $\mathsf{Sing}\, S$ are homeomorphic.*

2. Support theory for triangulated categories

This section is devoted to the study of support theory for triangulated categories. First of all, let us recall some basic notions from point-set topology and theory of triangulated categories.

Definition 3. Let X be a topological space and $\mathcal{T}$ a triangulated category.

(1) We say that X is *sober* if every irreducible closed subset of X is the closure of exactly one point.
(2) We say that X is *Noetherian* if every descending chain of closed subspaces stabilizes.
(3) We say that a subset W of X is *specialization-closed* if it is closed under specialization, namely if an element x of X belongs to W, then the closure $\overline{\{x\}}$ is contained in W. Note that W is specialization-closed if and only if it is a union of closed subspaces of X.
(4) We say that a non-empty additive full subcategory $\mathcal{X}$ of $\mathcal{T}$ is *thick* if it satisfies the following conditions:
   (a) closed under taking shifts: $\Sigma\mathcal{X} = \mathcal{X}$.
   (b) closed under taking extensions: for a triangle $L \to M \to N \to \Sigma L$ in $\mathcal{T}$, if L and N belong to $\mathcal{X}$, then so does M.
   (c) closed under taking direct summands: for two objects L, M of $\mathcal{T}$, if the direct sum $L \oplus M$ belongs to $\mathcal{X}$, then so do L and M.

   For a subcategory $\mathcal{X}$ of $\mathcal{T}$, denote by $\mathsf{thick}(\mathcal{X})$ the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$.

Next, let me introduce the notion of a support data which plays a central role in this article.

Definition 4. Let $\mathcal{T}$ be an essentially small triangulated category. A *support data* for $\mathcal{T}$ is a pair (X, σ) where X is a topological space and σ is an assignment which assigns to an object M of $\mathcal{T}$ a closed subset $\sigma(M)$ of X satisfying the following conditions:

(1) $\sigma(M) = \emptyset$ if and only if $M \cong 0$.
(2) $\sigma(\Sigma^n M) = \sigma(M)$ for any $M \in \mathcal{T}$ and any $n \in \mathbb{Z}$.
(3) $\sigma(M \oplus N) = \sigma(M) \cup \sigma(N)$ for any $M, N \in \mathcal{T}$.
(4) $\sigma(M) \subseteq \sigma(L) \cup \sigma(N)$ for any triangle $L \to M \to N \to \Sigma L$ in $\mathcal{T}$.

Support data naturally appear in various fields of mathematics.

Example 5. (1) Let X be a Noetherian scheme. Denote by $\mathsf{D}^{\mathsf{perf}}(X)$ the perfect derived category of X. Then the *cohomological support*

$$\mathsf{Supp}_X(M) := \{x \in X \mid M_x \ncong 0 \text{ in } \mathsf{D}^{\mathsf{perf}}(\mathcal{O}_{X,x})\}$$

defines a support data (X, Supp_X) for $\mathsf{D}^{\mathsf{perf}}(X)$.

(2) Let k be a field and G a finite group. Denote by $\underline{\mathsf{mod}}\, kG$ the stable category of finitely generated kG-module category. Then the *support variety* defines a support data $(\mathsf{Proj}\,\mathsf{H}^*(G;k), V_G)$ for $\underline{\mathsf{mod}}\, kG$ for details, please see [4].
(3) Let R be a commutative Noetherian ring R, the *singular support*

$$\mathsf{SSupp}_R(M) := \{\mathfrak{p} \in \mathsf{Sing}R \mid M_\mathfrak{p} \not\cong 0 \text{ in } \mathsf{D_{sg}}(R_\mathfrak{p})\}$$

defines a support data $(\mathsf{Sing}R, \mathsf{SSupp}_R)$ for $\mathsf{D_{sg}}(R)$.

Let (X, σ) be a support data for $\mathcal{T}$, $\mathcal{X}$ a thick subcategory of $\mathcal{T}$, and W a specialization closed subset of X. Then $f_\sigma(\mathcal{X}) := \bigcup_{M \in \mathcal{X}} \sigma(M)$ is a specialization closed subset of X and $g_\sigma(W) := \{M \in \mathcal{T} \mid \sigma(M) \subseteq W\}$ is a thick subcategory of $\mathcal{T}$. Therefore, we obtain two maps:

$$f_\sigma : \mathsf{Th}(\mathcal{T}) \rightleftarrows \mathsf{Spcl}(X) : g_\sigma.$$

Definition 6. Let (X, σ) be a support data for T. Then we say that (X, σ) is a *classifying support data for $\mathcal{T}$* if

(1) X is a Noetherian sober space, and
(2) the above maps f_σ and g_σ induces a one-to-one correspondence:

$$f_\sigma : \mathsf{Th}(\mathcal{T}) \rightleftarrows \mathsf{Spcl}(X) : g_\sigma.$$

The follwings are examples of classifying support data.

Example 7. (1) [16] Let X be a Noetherian quasi-affine scheme (i.e., an open subscheme of an affine scheme). Then the support data (X, Supp) for $\mathsf{D}^{\mathsf{perf}}(X)$ is classifying.
(2) [5, 6] Let k be a field and G a finite p-group. Then the support data $(\mathsf{Proj}\,\mathsf{H}^*(G;k), V_G)$ for $\underline{\mathsf{mod}}\, kG$ is classifying.

Let me give two more notions.

Definition 8. (1) We say that a thick subcategory $\mathcal{X}$ of $\mathcal{T}$ is *principal* if there is an object M of $\mathcal{T}$ such that $\mathcal{X} = \mathsf{thick}_\mathcal{T} M$. Denote by $\mathsf{PTh}(\mathcal{T})$ the set of all principal thick subcategories of $\mathcal{T}$.
(2) We say that a principal thick subcategory $\mathcal{X}$ of $\mathcal{T}$ is *irreducible* if $\mathcal{X} = \mathsf{thick}_\mathcal{T}(\mathcal{X}_1 \cup \mathcal{X}_2)$ $(\mathcal{X}_1, \mathcal{X}_2 \in \mathsf{PTh}(\mathcal{T}))$ implies that $\mathcal{X}_1 = \mathcal{X}$ or $\mathcal{X}_2 = \mathcal{X}$. Denote by $\mathsf{Irr}(\mathcal{T})$ the set of all irreducible thick subcategories of $\mathcal{T}$.

The following lemma shows that by using classifying support data, we can also classify principal thick subcategories and irreducible thick subcategories.

Lemma 9. *Let (X, σ) be a classifying support data for $\mathcal{T}$, then the one-to-one correspondence*

$$f_\sigma : \mathsf{Th}(\mathcal{T}) \rightleftarrows \mathsf{Spcl}(X) : g_\sigma.$$

restricts to one-to-one correspondences

$$f_\sigma : \mathsf{PTh}(\mathcal{T}) \rightleftarrows \mathsf{Cl}(X) : g_\sigma,$$

$$f_\sigma : \mathsf{Irr}(\mathcal{T}) \rightleftarrows \mathsf{Irr}(X) : g_\sigma.$$

Here, $\mathsf{Cl}(X)$ (resp. $\mathsf{Irr}(X)$) stands for the set of closed (resp. irreducible closed) subsets of X.

By using this lemma, we can show the uniqueness of classifying support data.

Theorem 10. *Let (X, σ) and (Y, σ) be classifying support data for essentially small triangulated categories $\mathcal{T}$ and $\mathcal{T}'$, respectively. If $\mathcal{T}$ and $\mathcal{T}'$ are equivalent as triangulated categories, then there is a homeomorphism $\varphi : X \to Y$ such that $\tau = \varphi \circ \sigma$.*

Outline of the proof. Let $F : \mathcal{T} \to \mathcal{T}'$ be a triangle equivalence. Then F induces a bijection $\tilde{F} : \mathsf{Th}(\mathcal{T}) \to \mathsf{Th}(\mathcal{T}')$ by the assignment

$$\mathsf{Th}(\mathcal{T}) \ni \mathcal{X} \mapsto \{N \in \mathcal{T}' \mid \exists M \in \mathcal{X} \text{ such that } N \cong F(M)\} \in \mathsf{Th}(\mathcal{T}').$$

For an object $M \in \mathcal{T}$, set $\tau^F(M) := \tau(F(M))$. Then one can easily verify that the pair (Y, τ^F) is a classifying support data for $\mathcal{T}$. Thus, we may assume $\mathcal{T}' = \mathcal{T}$.

Note that for a topological space X, the natural map $\iota_X : X \to \mathsf{Irr}(X), x \mapsto \overline{\{x\}}$ is bijective if and only if X is sober. Define maps $\varphi : X \to Y$ and $\psi : Y \to X$ to be the composites

$$\varphi : X \xrightarrow{\iota_X} \mathsf{Irr}(X) \xrightarrow{g_\sigma} \mathsf{Irr}(\mathcal{T}) \xrightarrow{f_\tau} \mathsf{Irr}(Y) \xrightarrow{\iota_Y^{-1}} Y,$$

$$\psi : Y \xrightarrow{\iota_Y} \mathsf{Irr}(Y) \xrightarrow{g_\tau} \mathsf{Irr}(\mathcal{T}) \xrightarrow{f_\sigma} \mathsf{Irr}(X) \xrightarrow{\iota_X^{-1}} X.$$

Then φ and ψ are well defined and mutually inverse bijections by Lemma 9. Furthermore, we can prove that the maps φ and ψ sends irreducible closed subsets to irreducible closed subsets as all maps f_σ, f_τ, g_σ and g_τ preserve inclusion relations. Thus, φ is a homeomorphism. $\square$

Applying this theorem to cases in Example 7, we have the following corollary.

Corollary 11. (1) *Let X and Y be Noetherian quasi-affine schemes. If $\mathsf{D}^{\mathsf{perf}}(X)$ and $\mathsf{D}^{\mathsf{perf}}(Y)$ are equivalent as triangulated categories. Then X and Y are homeomorphic. In particular, dimensions and numbers of irreducible components of X and Y are the same.*

(2) *Let k (resp l) be a field of characteristic p (resp. q) and G (resp. H) be a finite p-group (resp. q-group). If $\underline{\mathsf{mod}}\, kG$ and $\underline{\mathsf{mod}}\, lH$ are equivalent as triangulated categories, then $\mathsf{Proj}\, \mathsf{H}^*(G; k)$ and $\mathsf{Proj}\, \mathsf{H}^*(H; l)$ are homeomorphic. In particular, p-rank of G and q-rank of H are the same. Here, p-rank of a finite group G is the maximum rank of submodules of the form $(\mathbb{Z}/p\mathbb{Z})^r$.*

3. Singular equivalences

In this section, we give our main result and its applications.

We say that a Noetherian local ring $(R, \mathfrak{m}, k)$ is locally a hypersurface on the punctured spectrum if $R_\mathfrak{p}$ is a hypersurface for any prime $\mathfrak{p} \neq \mathfrak{m}$. To prove Theorem 2, we use the following result.

Theorem 12. [15] *Let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring which is hypersurface on the punctured spectrum. Then there is a one-to-one correspondence:*

$$\{\mathcal{X} \in \mathsf{Th}(\mathsf{D}_{\mathsf{sg}}(R)) \mid k \in \mathcal{X}\} \rightleftarrows \{W \in \mathsf{Spcl}(\mathsf{Sing}\, R) \mid W \neq \emptyset\}$$

From this bijection, we have a one-to-one correspondence

$$\mathsf{Th}(\mathsf{D_{sg}}(R)/\operatorname{\mathsf{thick}} k) \rightleftarrows \mathsf{Spcl}(\operatorname{\mathsf{Sing}} R \setminus \{\mathfrak{m}\})$$

and hence we obtain a classifying support data for $\mathsf{D_{sg}}(R)/\operatorname{\mathsf{thick}} k$ given by $\sigma(M) := \mathsf{SSupp(M)} \setminus \{\mathfrak{m}\}$ for $M \in \mathsf{D_{sg}}(R)/\operatorname{\mathsf{thick}} k$. We would like to apply Theorem 10 for this classifying support data. However, there is a problem that whether singularly equivalent between R and S implies triangle equivalence $\mathsf{D_{sg}}(R)/\operatorname{\mathsf{thick}} k \cong \mathsf{D_{sg}}(S)/\operatorname{\mathsf{thick}} l$.

To solve the problem, we use the following class of objects of a triangulated category.

Definition 13. Let $\mathcal{T}$ be a triangulated category. We say that $T \in \mathcal{T}$ is a *test object* if for any object M of $\mathcal{T}$,

$$\mathsf{Hom}_{\mathcal{T}}(T, \Sigma^n M) = 0 \text{ for } n \gg 0 \Rightarrow M = 0.$$

Remark 14. Test objects are preserved by triangle equivalences.

Next proposition is the key to prove Theorem 2.

Proposition 15. *Let $(R, \mathfrak{m}, k)$ be a complete intersection ring and T an object of $\mathsf{D_{sg}}(R)$. Then the following are equivalent:*

(1) *T is a test object of $\mathsf{D_{sg}}(R)$.*
(2) *$k \in \mathsf{thick}_{\mathsf{D_{sg}}(R)}(T)$.*

In particular, for a thick subcategory $\mathcal{X}$ of $\mathsf{D_{sg}}(R)$, $\mathcal{X}$ contains k if and only if $\mathcal{X}$ contains a test object of $\mathsf{D_{sg}}(R)$.

Corollary 16. *Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ be complete intersections. If R and S are singularly equivalent, then*

$$\mathsf{D_{sg}}(R)/\operatorname{\mathsf{thick}} k \cong \mathsf{D_{sg}}(S)/\operatorname{\mathsf{thick}} l$$

Splicing Theorem 10, Theorem 12 and Corollary 16, we obtain the following main result.

Theorem 17. *Let $(R, \mathfrak{m}, k)$ and $(S, \mathfrak{n}, l)$ be complete intersections which are locally hypersurfaces on the punctured spectra. If R and S are singularly equivalent, then $\operatorname{\mathsf{Sing}} R$ and $\operatorname{\mathsf{Sing}} S$ are homeomorphic.*

Let R be a commutative Noetherian ring and $\mathfrak{p}$ an element of $\operatorname{\mathsf{Sing}} R$. Then the full subcategory $\mathcal{X}_{\mathfrak{p}} := \{M \in \mathsf{D_{sg}}(R) \mid M_{\mathfrak{p}} \cong 0 \text{ in } \mathsf{D_{sg}}(R_{\mathfrak{p}})\}$ is a thick subcategory of $\mathsf{D_{sg}}(R)$. Since the localization functor $\mathsf{D_{sg}}(R) \to \mathsf{D_{sg}}(R_{\mathfrak{p}}), M \mapsto M_{\mathfrak{p}}$ is exact and $\mathcal{X}_{\mathfrak{p}}$ is its kernel, this functor induces an exact functor

$$\mathsf{D_{sg}}(R)/\mathcal{X}_{\mathfrak{p}} \to \mathsf{D_{sg}}(R_{\mathfrak{p}}).$$

Lemma 18. *Let R be a Gorenstein ring. Then the functor*

$$\mathsf{D_{sg}}(R)/\mathcal{X}_{\mathfrak{p}} \to \mathsf{D_{sg}}(R_{\mathfrak{p}})$$

is equivalent.

This lemma yields the following stronger result.

Corollary 19. *Let R and S be complete intersection rings which are locally hypersurfaces on the punctured spectra. If R and S are singularly equivalent, then there is a homeomorphism $\varphi : \operatorname{\mathsf{Sing}} R \to \operatorname{\mathsf{Sing}} S$ such that $R_{\mathfrak{p}}$ and $S_{\varphi(\mathfrak{p})}$ are singularly equivalent for any $\mathfrak{p} \in \operatorname{\mathsf{Sing}} R$.*

Next application says that Knörrer type periodicities fail over non-regular ring $S/(u^r)$.

Corollary 20. *Let S be a regular local ring, $r > 1$ an integer and f a non-zero element of S contained in the square of maximal ideal of S. Assume that $S/(f)$ has an isolated singularity. Then one has*

$$\mathsf{D}_{\mathsf{sg}}(S[[u]]/(f, u^r)) \not\cong \mathsf{D}_{\mathsf{sg}}(S[[u, v, w]]/(f + vw, u^r)).$$

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