ON FINITELY GRADED IG-ALGEBRAS AND THE STABLE CATEGORIES OF THEIR (GRADED) CM-MODULES

HIROYUKI MINAMOTO AND KOTA YAMAURA

Abstract. We discuss finitely graded Iwanaga-Gorenstein (IG) algebras A and representation theory of their (graded) Cohen-Macaulay (CM) modules. By quasi-Veronese algebra construction, in principle, we may reduce our study to the case where A is a trivial extension algebra $A = \Lambda \oplus C$ with the grading $\deg \Lambda = 0,\ \deg C = 1$. We give a necessary and sufficient condition that A is IG in terms of Λ and C by using derived tensor products and derived Homs. For simplicity, in the sequel, we assume that Λ is of finite global dimension. Then, we show that the condition that A is IG, has a triangulated categorical interpretation. We prove that if A is IG, then the graded stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ of CM-modules is realized as an admissible subcategory of the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$. As a corollary, we deduce that the Grothendieck group $K_0(\underline{\mathsf{CM}}^{\mathbb{Z}}A)$ is free of finite rank. We show that the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ of (non-graded) CM-modules is realized as the orbit category of the derived category $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ with respect to a certain autoequivalence.

We give several applications. Among other things, for a path algebra $\Lambda = KQ$ of an A_2 or A_3 quiver Q, we give a complete list of Λ-Λ-bimodule C such that $\Lambda \oplus C$ is IG by using the triangulated categorical interpretation mentioned above.

1. Introduction

This is a brief report on [6, 7], the main aim of which is a general study of representation theory of finitely graded Iwanaga-Gorenstein algebras.

Representation theory of (graded) Iwanaga-Gorenstein algebra was initiated by Auslander-Reiten [1], Happel [4] and Buchweitz [2], has been studied by many researchers and is recently getting interest from other areas.

Recall that an algebra A is called *Iwanaga-Gorenstein* (IG) if it is Noetherian and of finite injective dimension on both sides. A module M over an IG-algebra A is called *Cohen-Macaulay* (CM) if $\mathrm{Ext}^i_A(M, A) = 0$ for any $i > 0$. The full subcategory $\mathsf{CM}A \subset \mathrm{mod}A$ of CM-modules forms a Frobenius category such that the admissible projective-injective objects are precisely projective A-modules. Hence the stable category $\underline{\mathsf{CM}}^{\mathbb{Z}}A = \mathsf{CM}A/[\mathrm{proj}A]$ has a structure of triangulated categories. Representation theory of IG algebras mainly studies these categories.

For a Noetherian algebra A, the singular derived category $\mathrm{Sing}A$ is defined as the Verdier quotient $\mathrm{Sing}A := \mathsf{D}^{\mathrm{b}}(\mathrm{mod}A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}A)$. (If we perform the same construction to an algebraic variety X, then we obtain a triangulated category $\mathrm{Sing}X$ which only related to the singular locus of X. Hence, the name. This category plays an important role in Mirror symmetry.) Buchweitz and Happel showed that if A is IG, then there exists a canonical equivalence $\underline{\mathsf{CM}}^{\mathbb{Z}}A \cong \mathrm{Sing}A$.

The detailed version of this paper will be submitted for publication elsewhere.

We have the same story for a graded IG algebra and graded CM-modules over it.

1.1. **Remarks on generality.** In this proceeding, we restrict ourselves to deal with finite dimensional algebras over a field K. (Bi)modules are always finite dimensional and bimodule is always K-central. Moreover, we often give our result under the assumption that $\mathrm{gldim}\Lambda < \infty$. However almost of all results are verified in more or less wider generality.

For the general form of Theorem 2,3,4,6,7 we refer [6]. Theorem 7,8 are verified for a (not necessarily finite dimensional) IG-algebra Λ. But we need to replace $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ with $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$. We also need to replace $\underline{\mathsf{CM}}^{\mathbb{Z}}A$ with the stable category of "locally perfect" CM-modules, the definition of which will be given our forthcoming paper [7].

2. Quasi-Veronese algebra construction

We recall quasi-Veronese algebra construction introduced by Mori [8].

Let $B = \bigoplus_{i\in\mathbb{Z}} B_i$ be a $\mathbb{Z}$-graded algebra. For $e \in \mathbb{N}$, we define the e-th quasi-Veronese algebra $B^{[e]}$ of B as below

$$B^{[e]} := \bigoplus_{i\in\mathbb{Z}} B_i^{[e]}, \quad B_i^{[e]} := \begin{pmatrix} B_{ei} & B_{ei+1} & \cdots & B_{e(i+1)-1} \\ B_{ei-1} & B_{ei} & \cdots & B_{e(i+1)-2} \\ \vdots & \vdots & & \vdots \\ B_{e(i-1)+1} & B_{e(i-1)+2} & \cdots & B_{ei} \end{pmatrix}$$

The i-th degree part is $B_i^{[e]}$ and the multiplication is the matrix multiplication. The basic fact is the following.

Theorem 1. *B and $B^{[e]}$ are graded Morita equivalent to each other.*

$$\mathsf{qv} : \mathrm{GrMod}B \cong \mathrm{GrMod}B^{[e]}$$

It is may helpful for understanding to pointing out that $B^{[e]}$ is nothing but the endomorphism algebra of $G = B \oplus B(-1) \oplus \cdots \oplus B(-e+1)$ with the grading induced from the e-th degree shift functor (e).

$$B^{[e]} \cong \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{GrMod}B}(G, G(ie))$$

We focus our attention to a finitely graded algebra $A = \bigoplus_{i=0}^{\ell} A_i$. Then an easy but helpful observation is that ℓ-th quasi-Veronese algebra $A^{[\ell]}$ is concentrated in degree 0 and 1. If we set

$$\nabla A := A_0^{[\ell]} = \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}, \quad \Delta A := A_1^{[\ell]} = \begin{pmatrix} A_\ell & 0 & \cdots & 0 \\ A_{\ell-1} & A_\ell & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \cdots & A_\ell \end{pmatrix},$$

then $A^{[\ell]}$ is the trivial extension algebra of ∇A by ΔA with the canonical grading $\deg \nabla A = 0$ and $\deg \Delta A = 1$.

$$A^{[\ell]} = \nabla A \oplus \Delta A$$

We note that the algebra ∇A is called the Beilinson algebra of A.

In view of Theorem 1, A and $A^{[\ell]}$ share every representation theoretic property. In particular, A is IG if and only if so is $A^{[\ell]}$. If this is the case, then the equivalence qv induces an equivalence

$$\underline{\mathsf{CM}}^{\mathbb{Z}} A \cong \underline{\mathsf{CM}}^{\mathbb{Z}} A^{[\ell]}.$$

Hence, representation theoretic study of finitely graded IG-algebras can be, in principle, reduced to IG-algebra which is a trivial extension algebra $A = \Lambda \oplus C$.

3. When is $A = \Lambda \oplus C$ IG? When A is IG!

We investigate the question posed in the section title. An iterated derived tensor product C^a plays a key role.

$$C^a := C \otimes^{\mathbb{L}}_{\Lambda} C \otimes^{\mathbb{L}}_{\Lambda} \cdots \otimes^{\mathbb{L}}_{\Lambda} C \quad (a\text{-factors})$$

First we study the following related question.

3.1. **When gldim $A < \infty$?** The following is essentially proved by Palmer-Roos [10] and Löfwall [5].

Theorem 2. $\mathrm{gldim} A < \infty$ *if and only if* $\mathrm{gldim}\Lambda < \infty$ *and* $C^a = 0$ *for* $a \gg 0$.

For the proof we make use of the canonical grading of A, namely $\deg \Lambda = 0$ and $\deg C = 1$. Orlov [9] introduced a decomposition of complexes of graded projective A-module according to the degree of generators. By closely looking the decomposition, we obtain the above result. For the details we refer [6]. By the same method, we get a description of the kernel of the canonical functor

$$\varpi : \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda) \xrightarrow{\text{deg 0 embed.}} \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} A) \xrightarrow{\text{quotient}} \mathrm{Sing}^{\mathbb{Z}} A = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}} A)/\mathsf{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}} A)$$

where the first functor regard a complex M of Λ-modules as a complex of graded A-modules concentrated in 0-th degree.

Theorem 3. *Assume that C has finite projective dimension as a right Λ-module. Then*

$$\mathrm{Ker}\varpi = \bigcup_{a \geq 0} \mathrm{Ker}(- \otimes^{\mathbb{L}} C^a)|_{\mathsf{K}}$$

3.2. **When id$A < \infty$?** Let λ^a_r be the morphism below induced from $- \otimes^{\mathbb{L}}_{\Lambda} C$

$$\lambda^a_r : \mathbb{R}\mathrm{Hom}_{\Lambda}(C^a, \Lambda) \to \mathbb{R}\mathrm{Hom}_{\Lambda}(C^{a+1}, C)$$

where the subscript r stands for "right".

Theorem 4. *Assume that* $\mathrm{gldim}\Lambda < \infty$. *Then* $\mathrm{id} A < \infty$ *if and only if the morphism* λ^a_r *is an isomorphism in* $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ *for* $a \gg 0$.

We call the latter condition the *right asid* condition, where asid is abbreviation of *attaching self-injective dimension.*

We introduce an important invariant for a right asid module.

Definition 5. Assume that C satisfies the right asid condition. Then, we define the *right asid number* α_r to be

$$\alpha_r := \min\{a \geq 0 \mid \lambda^a_r \text{ is an isomorphism.}\}.$$

This number relates to a graded minimal injective resolution $I^\bullet$ of A as in the following way.

Theorem 6. *Assume* $\mathrm{id}A < \infty$. *Let* $\Omega^{-n}A = \mathrm{Ker}[\delta^n : I^n \to I^{n+1}]$ *be the n-th cosyzygy. Then,*

$$\alpha_r = \max\{a \geq 1 \mid \exists n,\ \mathrm{soc}(\Omega^{-n}A)_{-a} \neq 0\} + 1.$$

3.3. **When is $A = \Lambda \oplus C$ IG?** Now it is easy to answer the question. Let λ_ℓ^a the left version of λ_r^a.

$$\lambda_\ell^a : \mathbb{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(C^a, \Lambda) \to \mathbb{R}\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(C^{a+1}, C)$$

Theorem 7. *Assume that* $\mathrm{gldim}\Lambda < \infty$. *Then $A = \Lambda \oplus C$ is IG if and only if the morphism λ_r^a and λ_ℓ^a are isomorphism for $a \gg 0$.*

We call an bimodule C asid if $A = \Lambda \oplus C$ is IG. For such module we can define the left asid number α_ℓ as well as the right asid number α_r.

$$\alpha_r := \min\{a \geq 0 \mid \lambda_r^a \text{ is an isomorphism.}\},$$
$$\alpha_\ell := \min\{a \geq 0 \mid \lambda_\ell^a \text{ is an isomorphism.}\}.$$

3.4. **Categorical characterization of asid bimodule.** The condition that C is asid bimodule has a characterization in a triangulated categorical term.

We recall that a subcategory E of a triangulated category D is called *admissible* if the canonical inclusion $\mathsf{E} \subset \mathsf{D}$ has a left adjoint functor and a right adjoint functor. It is known that E is admissible if and only if it fits the following two semi-orthogonal decompositions

$$\mathsf{D} = \mathsf{E} \perp \mathsf{E}^\perp = {}^\perp\mathsf{E} \perp \mathsf{E}.$$

Theorem 8. *Assume that* $\mathrm{gldim}\Lambda < \infty$. *A bimodule C over Λ is asid if and only if there exists an admissible subcategory $\mathsf{T} \subset \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ which satisfies the following conditions*

(1) *The functor $T = - \otimes_\Lambda^{\mathbb{L}} C$ acts on T as an equivalence, i.e., $T(\mathsf{T}) \subset \mathsf{T}$ and the restriction functor $T|_{\mathsf{T}} : \mathsf{T} \xrightarrow{\sim} \mathsf{T}$ is an equivalence.*

(2) *The functor $T = - \otimes_\Lambda^{\mathbb{L}} C$ nilpotently acts on $\mathsf{T}^\perp$, i.e., $T(\mathsf{T}^\perp) \subset \mathsf{T}^\perp$ and $T^a(\mathsf{T}^\perp) = 0$ for some $a \in \mathbb{N}$.*

3.5. **When $A = \Lambda \oplus C$ is IG!** When $A = \Lambda \oplus C$ is IG, we have the following result.

Theorem 9. *Assume that* $\mathrm{gldim}\Lambda < \infty$. *Let C be an asid bimodule over Λ. Then the followings hold.*

(1) $\alpha_r = \alpha_\ell$.

We put $\alpha := \alpha_r = \alpha_\ell$.

(2) *The admissible subcategory $\mathsf{T} \subset \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ satisfying the conditions (1) and (2) of Theorem 8 is uniquely determined as in the first equality below. The functor ϖ induces an equivalence shown as below.*

$$\mathsf{T} = \mathsf{thick}C^\alpha \overset{\varpi}{\cong} \underline{\mathsf{CM}}^{\mathbb{Z}}A.$$

(3) *The following equalities hold.*

$$\mathsf{T}^\perp = \mathrm{Ker}(- \otimes_\Lambda^{\mathbb{L}} C^\alpha) = \mathrm{Ker}\varpi$$

(4) $\alpha = \min\{a \geq 0 \mid \mathsf{T}^\perp \otimes^{\mathbb{L}} C^a = 0\}$.

We would like to mention one thing. A semi-orthogonal decomposition of a triangulated category is considered as a categorification of a direct sum decomposition of a vector space. Since $\mathsf{thick} C^{\alpha}$ can be considered as $\mathrm{Im}(- \otimes^{\mathbb{L}} C^{\alpha})$, thus, from the above view point, the semi-orthogonal decomposition of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ by T and $\mathsf{T}^{\perp}$ given in the above theorem can be looked as a categorification of a direct sum decomposition appeared in Fitting Lemma

$$\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda) = \mathrm{Im}(- \otimes^{\mathbb{L}}_{\Lambda} C^{\alpha}) \perp \mathrm{Ker}(- \otimes^{\mathbb{L}}_{\Lambda} C^{\alpha}).$$

3.6. **Application to a finitely graded IG-algebra.** By quasi-Veronese algebra construction, we deduce the following consequence from Theorem 9.

Corollary 10. *Let $A = \bigoplus_{i=0}^{\ell} A_i$ be a finitely graded IG-algebra. Assume that* $\mathrm{gldim} A_0 < \infty$. *Then the Grothendick group $K_0(\underline{\mathsf{CM}}^{\mathbb{Z}} A)$ is free of finite rank. Moreover,*

$$\mathrm{rank} K_0(\underline{\mathsf{CM}}^{\mathbb{Z}} A) \leq \ell |A|$$

where $|A|$ denotes the number of non-isomorphic simple A-modules.

This result follows from that the category $\underline{\mathsf{CM}}^{\mathbb{Z}} A \cong \underline{\mathsf{CM}}^{\mathbb{Z}} A^{[\ell]}$ is an admissible subcategory of $\mathsf{D}^{\mathrm{b}}(\mathrm{mod}\nabla A)$. Now it is clear the bound of the rank is nothing but the number of non-isomorphic simple ∇A-module.

4. Applications

4.1. **Two classes of CM-finite algebras.** As an application, we give two classes of CM-finite algebras. The main tool other than our result is the following theorem obtained in a joint work with M. Yoshiwaki, which is a CM-version of Gabriel's theorem which assert that finiteness of representation type is preserved by taking orbit category.

Theorem 11 (MY-Yoshiwaki)**.** *Let A be a finite dimensional graded IG algebra. Then, A is of finite CM type if and only if it is of finite graded CM type. Moreover, if this is the case, the functor* $\mathrm{mod}^{\mathbb{Z}} A \to \mathrm{mod} A$ *which forgets the grading induces the equality* $\mathrm{ind}\mathsf{CM}^{\mathbb{Z}} A/(1) = \mathrm{ind}\mathsf{CM} A$.

The first application is the followings. It is worth noting that the algebras A in the theorem below is possibly of infinite representation type.

Theorem 12. *Let Λ be an iterated tilted algebra of Dynkin type. If a trivial extension algebra $A = \Lambda \oplus C$ is IG, then it is of finite CM type.*

In the above theorem, CM-representation type is controlled by the degree 0-part. Contrary to this, in the next example, CM-representation theory is controlled by the degree 1-part.

An easy way to get a bimodule is to take a tensor product $N \otimes_K M$ of a right module N and a left module M.

Theorem 13. *Assume* $\mathrm{gldim}\Lambda < \infty$. *Let $A = \Lambda \oplus (N \otimes_K M)$. Then,*

1. $\mathrm{gldim} A < \infty$ *if and only if* $M \otimes^{\mathbb{L}}_{\Lambda} N = 0$.
2. *A is IG and* $\mathrm{gldim} A = \infty$ *if and only if* $\mathbb{R}\mathrm{Hom}(M, M) \cong K$ *and* $\mathbb{R}\mathrm{Hom}(M, \Lambda) = N[-p]$ *for some $p \in \mathbb{N}$.*

If (2) is the case, then the followings hold.

(a) *Let p be the integer in (2). Then* $p = \mathrm{pd}_\Lambda M = \mathrm{pd}_{\Lambda^{\mathrm{op}}} N$.
(b) $\underline{\mathsf{CM}}^{\mathbb{Z}} A \cong \mathsf{D}^{\mathrm{b}}(\mathrm{mod} K)$ *under which* (1) *corresponds* $[p+1]$.
(c) $\underline{\mathsf{CM}}^{\mathbb{Z}} A \cong (\mathrm{mod} K)^{\oplus p+1}$.
(d) $\mathrm{ind}\underline{\mathsf{CM}}^{\mathbb{Z}} A = \{M, \Omega M, \cdots, \Omega^p M\}$ *where the syzygyies are taken as A-modules.*

Example 14. Let Λ be a basic finite dimensional algebra of finite global dimension and $e, f \in \Lambda$ idempotent elements. Then the algebra $A = \Lambda \oplus (\Lambda e \otimes_K f\Lambda)$ is of finite global dimension if and only if $f\Lambda e = 0$. The algebra A is an IG algebra of infinite global dimension if and only if $e = f$ and $\dim e\Lambda e = 1$.

On the other hands, X-W. Chen [3] showed that $\mathrm{Sing}A$ is Hom-finite if and only if $\dim f\Lambda e \leq 1$. Thus we conclude that there are finite dimensional algebras A which is not IG but whose singular derived category $\mathrm{Sing}A$ is Hom-finite.

4.2. **Classification.** Using the categorical characterization of Theorem 8, we obtain the complete list of asid modules C when Λ is the path algebra of A_2-quiver or A_3-quiver in the following strategy.

Step 1. Classify admissible subcategories T of $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$.
Step 2. For an admissible subcategory T, classify bimodules C such that the functor $- \otimes^{\mathbb{L}}_\Lambda C$ acts on T as an equivalence and nilpotently acts on $\mathsf{T}^\perp$.

We give the list of T and C over $\Lambda = K[1 \leftarrow 2]$. In the list, P_1, P_2 denote the indecomposable projective modules which correspond to the vertex $1, 2$ respectively. I_2 is the indecomposable injective module which corresponds to the vertex 2. S_1^{left} denotes the simple left Λ-module which corresponds to the vertex 1. S_2^{right} denotes the simple right Λ-module which corresponds to the vertex 2.

(I) $\mathsf{T} = \mathsf{D}^{\mathrm{b}}(\mathrm{mod}\Lambda)$ (precisely the case $\alpha = 0$).
$C = \Lambda, D(\Lambda)$.
(II) $\mathsf{T} = \mathrm{add}\{P_1[i] \mid i \in \mathbb{Z}\}$
$C = \Lambda e_1 \otimes_K e_1\Lambda$
(III) $\mathsf{T} = \mathrm{add}\{P_2[i] \mid i \in \mathbb{Z}\}$
$C = \Lambda e_2 \otimes_K e_2\Lambda$
(IV) $\mathsf{T} = \mathrm{add}\{I_2[i] \mid i \in \mathbb{Z}\}$
$C = S_1^{\mathrm{left}} \otimes_K S_2^{\mathrm{right}}$
(V) $\mathsf{T} = 0$ (precisely the case $\mathrm{gldim}A < \infty$.)
(V-1) $C = (\Lambda e_2 \otimes_K e_1\Lambda)^{\oplus n}$
(V-2) $C = (S_1^{\mathrm{left}} \otimes_K e_2\Lambda)^{\oplus n}$
(V-3) $C = (\Lambda e_1 \otimes_K S_2^{\mathrm{right}})^{\oplus n}$

For the list of A_3 case, we refer [7].

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Hiroyuki Minamoto Department of Mathematics and Information Sciences Osaka Prefecture University Sakai Nakamozu, Osaka 599-8531 JAPAN *Email*: minamoto@mi.s.osakafu-u.ac.jp

Kota Yamaura Graduate Faculty of Interdisciplinary Research, Faculty of Engineering University of Yamanashi Kofu Takeda, Yamanashi 400-8510 JAPAN *Email*: kyamaura@yamanashi.ac.jp