ON FINITELY GRADED IG-ALGEBRAS AND THE STABLE CATEGORIES OF THEIR (GRADED) CM-MODULES

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Abstract. We discuss finitely graded Iwanaga-Gorenstein (IG) algebras $A$ and representation theory of their (graded) Cohen-Macaulay (CM) modules. By quasi-Veronese algebra construction, in principle, we may reduce our study to the case where $A$ is a trivial extension algebra $A = \Lambda \oplus C$ with the grading $\deg \Lambda = 0$, $\deg C = 1$. We give a necessary and sufficient condition that $A$ is IG in terms of $\Lambda$ and $C$ by using derived tensor products and derived Hom's. For simplicity, in the sequel, we assume that $\Lambda$ is of finite global dimension. Then, we show that the condition that $A$ is IG, has a triangulated categorical interpretation. We prove that if $A$ is IG, then the graded stable category $\text{CM}^Z A$ of CM-modules is realized as an admissible subcategory of the derived category $\mathcal{D}b(\text{mod}\Lambda)$. As a corollary, we deduce that the Grothendieck group $K_0(\text{CM}^Z A)$ is finite rank. We show that the stable category $\text{CM}^Z A$ of (non-graded) CM-modules is realized as the orbit category of the derived category $\mathcal{D}b(\text{mod}\Lambda)$ with respect to a certain autoequivalence.

We give several applications. Among other things, for a path algebra $\Lambda = KQ$ of an $A_2$ or $A_3$ quiver $Q$, we give a complete list of $\Lambda$-$\Lambda$-bimodule $C$ such that $\Lambda \oplus C$ is IG by using the triangulated categorical interpretation mentioned above.

1. Introduction

This is a brief report on [6, 7], the main aim of which is a general study of representation theory of finitely graded Iwanaga-Gorenstein algebras.

Representation theory of (graded) Iwanaga-Gorenstein algebra was initiated by Auslander-Reiten [1], Happel [4] and Buchweitz [2], has been studied by many researchers and is recently getting interest from other areas.

Recall that an algebra $A$ is called Iwanaga-Gorenstein (IG) if it is Noetherian and of finite injective dimension on both sides. A module $M$ over an IG-algebra $A$ is called Cohen-Macaulay (CM) if $\text{Ext}_A^i(M, A) = 0$ for any $i > 0$. The full subcategory $\text{CMA} \subset \text{mod}\Lambda$ of CM-modules forms a Frobenius category such that the admissible projective-injective objects are precisely projective $A$-modules. Hence the stable category $\text{CM}^Z A = \text{CMA}/[\text{proj}\Lambda]$ has a structure of triangulated categories. Representation theory of IG algebras mainly studies these categories.

For a Noetherian algebra $A$, the singular derived category $\text{Sing}A$ is defined as the Verdier quotient $\text{Sing}A := \mathcal{D}b(\text{mod}\Lambda)/\mathcal{K}b(\text{proj}\Lambda)$. (If we perform the same construction to an algebraic variety $X$, then we obtain a triangulated category $\text{Sing}X$ which only related to the singular locus of $X$. Hence, the name. This category plays an important role in Mirror symmetry.) Buchweitz and Happel showed that if $A$ is IG, then there exists a canonical equivalence $\text{CM}^Z A \cong \text{Sing}A$.

The detailed version of this paper will be submitted for publication elsewhere.
We have the same story for a graded IG algebra and graded CM-modules over it.

1.1. Remarks on generality. In this proceeding, we restrict ourselves to deal with finite dimensional algebras over a field \( K \). (Bi)modules are always finite dimensional and bimodule is always \( K \)-central. Moreover, we often give our result under the assumption that \( \text{gldim} \Lambda < \infty \). However almost of all results are verified in more or less wider generality.

For the general form of Theorem 2,3,4,6,7 we refer [6]. Theorem 7,8 are verified for a \((\text{not necessarily finite dimensional})\) IG-algebra \( \Lambda \). But we need to replace \( D^b(\text{mod}\Lambda) \) with \( K^b(\text{proj}\Lambda) \). We also need to replace \( \text{CM}Z^{}A \) with the stable category of “locally perfect” CM-modules, the definition of which will be given our forthcoming paper [7].

2. Quasi-Veronese algebra construction

We recall quasi-Veronese algebra construction introduced by Mori [8].

Let \( B = \bigoplus_{i \in \mathbb{Z}} B_i \) be a \( \mathbb{Z} \)-graded algebra. For \( e \in \mathbb{N} \), we define the \( e \)-th quasi-Veronese algebra \( B[e] \) of \( B \) as below

\[
B[e] := \bigoplus_{i \in \mathbb{Z}} B_i^e, \quad B_i^e := \left( \begin{array}{cccc}
B_{ei} & B_{ei+1} & \cdots & B_{e(i+1)-1} \\
B_{ei-1} & B_{ei} & \cdots & B_{e(i+1)-2} \\
& \vdots & \ddots & \vdots \\
B_{e(i-1)+1} & B_{e(i-1)+2} & \cdots & B_{ei}
\end{array} \right)
\]

The \( i \)-th degree part is \( B_i^e \) and the multiplication is the matrix multiplication. The basic fact is the following.

**Theorem 1.** \( B \) and \( B[e] \) are graded Morita equivalent to each other.

\[ qv : \text{GrMod} B \cong \text{GrMod} B[e] \]

It is may helpful for understanding to pointing out that \( B[e] \) is nothing but the endomorphism algebra of \( G = B \oplus B(-1) \oplus \cdots \oplus B(-e+1) \) with the grading induced from the \( e \)-th degree shift functor \((e)\).

\[ B[e] \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\text{GrMod} B}(G, G(ie)) \]

We focus our attention to a finitely graded algebra \( A = \bigoplus_{i=0}^\ell A_i \). Then an easy but helpful observation is that \( \ell \)-th quasi-Veronese algebra \( A[\ell] \) is concentrated in degree 0 and 1. If we set

\[
\nabla A := A[0]^{\ell} = \left( \begin{array}{cccc}
A_0 & A_1 & \cdots & A_{\ell-1} \\
0 & A_0 & \cdots & A_{\ell-2} \\
& \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_0
\end{array} \right), \quad \Delta A := A[1]^{\ell} = \left( \begin{array}{cccc}
A_0 & 0 & \cdots & 0 \\
A_{\ell-1} & A_0 & \cdots & 0 \\
& \vdots & \ddots & \vdots \\
A_1 & A_2 & \cdots & A_\ell
\end{array} \right),
\]

then \( A[\ell] \) is the trivial extension algebra of \( \nabla A \) by \( \Delta A \) with the canonical grading \( \text{deg} \nabla A = 0 \) and \( \text{deg} \Delta A = 1 \).

\[ A[\ell] = \nabla A \oplus \Delta A \]

We note that the algebra \( \nabla A \) is called the Beilinson algebra of \( A \).

2
In view of Theorem 1, $A$ and $A^{[\ell]}$ share every representation theoretic property. In particular, $A$ is IG if and only if so is $A^{[\ell]}$. If this is the case, then the equivalence $qv$ induces an equivalence 

\[ \mathcal{CM}^\mathbb{Z} A \cong \mathcal{CM}^\mathbb{Z} A^{[\ell]} \]

Hence, representation theoretic study of finitely graded IG-algebras can be, in principle, reduced to IG-algebra which is a trivial extension algebra $A = \Lambda \oplus C$.

3. When is $A = \Lambda \oplus C$ IG? When $A$ is IG!

We investigate the question posed in the section title. An iterated derived tensor product $C^a$ plays a key role.

First we study the following related question.

3.1. When $\text{gldim} A < \infty$? The following is essentially proved by Palmer-Roos [10] and Lofwall [5].

**Theorem 2.** $\text{gldim} A < \infty$ if and only if $\text{gldim} A < \infty$ and $C^a = 0$ for $a \gg 0$.

For the proof we make use of the canonical grading of $A$, namely $\text{deg} \Lambda = 0$ and $\text{deg} C = 1$. Orlov [9] introduced a decomposition of complexes of graded projective $A$-module according to the degree of generators. By closely looking the decomposition, we obtain the above result. For the details we refer [6]. By the same method, we get a description of the kernel of the canonical functor

\[ \varpi : D^b(\text{mod} \Lambda) \xrightarrow{\text{deg} 0 \text{ embed.}} D^b(\text{mod} \Lambda) \xrightarrow{\text{quotient}} \text{Sing} \mathbb{Z} A = D^b(\text{mod} \Lambda) / K^b(\text{proj} \mathbb{Z} A) \]

where the first functor regard a complex $M$ of $\Lambda$-modules as a complex of graded $A$-modules concentrated in 0-th degree.

**Theorem 3.** Assume that $C$ has finite projective dimension as a right $\Lambda$-module. Then

\[ \text{Ker} \varpi = \bigcup_{a \geq 0} \text{Ker}(- \otimes^\mathbb{L} A^a) |_K \]

3.2. When $\text{id} A < \infty$? Let $\lambda^a_r$ be the morphism below induced from $- \otimes^\mathbb{L} A^a$

\[ \lambda^a_r : \mathbb{R} \text{Hom}_\Lambda(C^a, \Lambda) \rightarrow \mathbb{R} \text{Hom}_\Lambda(C^{a+1}, C) \]

where the subscript $r$ stands for “right”.

**Theorem 4.** Assume that $\text{gldim} \Lambda < \infty$. Then $\text{id} A < \infty$ if and only if the morphism $\lambda^a_r$ is an isomorphism in $D^b(\text{mod} \Lambda)$ for $a \gg 0$.

We call the latter condition the right asid condition, where asid is abbreviation of attaching self-injective dimension.

We introduce an important invariant for a right asid module.

**Definition 5.** Assume that $C$ satisfies the right asid condition. Then, we define the right asid number $\alpha_r$ to be

\[ \alpha_r := \min\{a \geq 0 \mid \lambda^a_r \text{ is an isomorphism.}\} \]
This number relates to a graded minimal injective resolution \( I^* \) of \( A \) as in the following way.

**Theorem 6.** Assume \( \text{id}A < \infty \). Let \( \Omega^{-n}A = \text{Ker}[\delta^n : I^n \to I^{n+1}] \) be the \( n \)-th cosyzygy. Then,

\[ \alpha_r = \max\{a \geq 1 \mid \exists n, \, \text{soc}(\Omega^{-n}A)_{-a} \neq 0\} + 1. \]

3.3. **When is \( A = \Lambda \oplus C \) IG?** Now it is easy to answer the question. Let \( \lambda^0 \) the left version of \( \lambda_a \).

\[ \lambda^0 : \mathbb{R}\text{Hom}_{\Lambda^{\text{op}}}(C^a, \Lambda) \to \mathbb{R}\text{Hom}_{\Lambda^{\text{op}}}(C^{a+1}, C) \]

**Theorem 7.** Assume that \( \text{gldim} \Lambda < \infty \). Then \( A = \Lambda \oplus C \) is IG if and only if the morphism \( \lambda^0 \) and \( \lambda^0 \) are isomorphism for \( a \gg 0 \).

We call an bimodule \( C \) asid if \( A = \Lambda \oplus C \) is IG. For such module we can define the left asid number \( \alpha_{\ell} \) as well as the right asid number \( \alpha_r \).

\[ \alpha_r := \min\{a \geq 0 \mid \lambda^0 \text{ is an isomorphism.}\}, \]
\[ \alpha_{\ell} := \min\{a \geq 0 \mid \lambda^0 \text{ is an isomorphism.}\}. \]

3.4. **Categorical characterization of asid bimodule.** The condition that \( C \) is asid bimodule has a characterization in a triangulated categorical term.

We recall that a subcategory \( E \) of a triangulated category \( D \) is called **admissible** if the canonical inclusion \( E \subset D \) has a left adjoint functor and a right adjoint functor. It is known that \( E \) is admissible if and only if it fits the following two semi-orthogonal decompositions

\[ D = E \perp E^\perp = {^+E} \perp E. \]

**Theorem 8.** Assume that \( \text{gldim} \Lambda < \infty \). A bimodule \( C \) over \( \Lambda \) is asid if and only if there exists an admissible subcategory \( T \subset D^b(\text{mod} \Lambda) \) which satisfies the following conditions

1. **The functor** \( T = - \otimes^L \Lambda C \) **acts on** \( T \) **as an equivalence**, i.e., \( T(T) \subset T \) and the restriction functor \( T|_T : T \twoheadrightarrow T \) is an equivalence.

2. **The functor** \( T = - \otimes^L \Lambda C \) **nilpotently acts on** \( T^\perp \), i.e., \( T(T^\perp) \subset T^\perp \) and \( T^a(T^\perp) = 0 \) for some \( a \in \mathbb{N} \).

3.5. **When \( A = \Lambda \oplus C \) is IG!** When \( A = \Lambda \oplus C \) is IG, we have the following result.

**Theorem 9.** Assume that \( \text{gldim} \Lambda < \infty \). Let \( C \) be an asid bimodule over \( \Lambda \). Then the followings hold.

1. \( \alpha_r = \alpha_{\ell} \).

We put \( \alpha := \alpha_r = \alpha_{\ell} \).

2. **The admissible subcategory** \( T \subset D^b(\text{mod} \Lambda) \) **satisfying the conditions (1) and (2) of Theorem 8** is uniquely determined as in the first equality below. The functor \( \varpi \) induces an equivalence shown as below.

\[ T = \text{thick}C^a \cong \text{CM}^Z A. \]

3. **The following equalities hold.**

\[ T^\perp = \text{Ker}( - \otimes^L \Lambda C^a ) = \text{Ker} \varpi \]

4. \( \alpha = \min\{a \geq 0 \mid T^\perp \otimes^L C^a = 0\} \).
We would like to mention one thing. A semi-orthogonal decomposition of a triangulated category is considered as a categorification of a direct sum decomposition of a vector space. Since \( \text{thick} \mathcal{C} \) can be considered as \( \text{Im}(L \mathcal{C}) \), thus, from the above viewpoint, the semi-orthogonal decomposition of \( \mathcal{D} \text{b}(\text{mod}) \) by \( T \) and \( T^\perp \) given in the above theorem can be looked as a categorification of a direct sum decomposition appeared in Fitting Lemma

\[
\mathcal{D} \text{b}(\text{mod}) = \text{Im}(- \otimes^L_\mathcal{C} C^\alpha) \perp \text{Ker}(- \otimes^L_\mathcal{C} C^\alpha).
\]

3.6. Application to a finitely graded IG-algebra. By quasi-Veronese algebra construction, we deduce the following consequence from Theorem 9.

Corollary 10. Let \( A = \bigoplus_{i=0}^\ell A_i \) be a finitely graded IG-algebra. Assume that \( \text{gldim} A_0 < \infty \). Then the Grothendieck group \( K_0(\mathcal{C} \text{M}^Z A) \) is free of finite rank. Moreover,

\[
\text{rank} K_0(\mathcal{C} \text{M}^Z A) \leq \ell |A|
\]

where \( |A| \) denotes the number of non-isomorphic simple \( A \)-modules.

This result follows from that the category \( \mathcal{C} \text{M}^Z A \cong \mathcal{C} \text{M}^Z A[\ell] \) is an admissible subcategory of \( \mathcal{D} \text{b}(\text{mod} \nabla A) \). Now it is clear the bound of the rank is nothing but the number of non-isomorphic simple \( \nabla A \)-module.

4. Applications

4.1. Two classes of CM-finite algebras. As an application, we give two classes of CM-finite algebras. The main tool other than our result is the following theorem obtained in a joint work with M. Yoshiwaki, which is a CM-version of Gabriel’s theorem which assert that finiteness of representation type is preserved by taking orbit category.

Theorem 11 (MY-Yoshiwaki). Let \( A \) be a finite dimensional graded IG algebra. Then, \( A \) is of finite CM type if and only if it is of finite graded CM type. Moreover, if this is the case, the functor \( \text{mod}^Z A \to \text{mod} A \) which forgets the grading induces the equality \( \text{indCM}^Z A/(1) = \text{indCM} A \).

The first application is the followings. It is worth noting that the algebras \( A \) in the theorem below is possibly of infinite representation type.

Theorem 12. Let \( \Lambda \) be an iterated tilted algebra of Dynkin type. If a trivial extension algebra \( A = \Lambda \oplus C \) is IG, then it is of finite CM type.

In the above theorem, CM-representation type is controlled by the degree 0-part. Contrary to this, in the next example, CM-representation theory is controlled by the degree 1-part.

An easy way to get a bimodule is to take a tensor product \( N \otimes_K M \) of a right module \( N \) and a left module \( M \).

Theorem 13. Assume \( \text{gldim} \Lambda < \infty \). Let \( A = \Lambda \oplus (N \otimes_K M) \). Then,

1. \( \text{gldim} A < \infty \) if and only if \( M \otimes_\Lambda^R N = 0 \).
2. \( A \) is IG and \( \text{gldim} A = \infty \) if and only if \( \text{RHom}(M, M) \cong K \) and \( \text{RHom}(M, \Lambda) = N[-p] \) for some \( p \in \mathbb{N} \).

If (2) is the case, then the followings hold.
(a) Let \( p \) be the integer in (2). Then \( p = \text{pd}_\Lambda M = \text{pd}_{\Lambda^{op}} N \).

(b) \( \text{CM}^Z A \cong D^b(\text{mod}K) \) under which (1) corresponds \( [p + 1] \).

(c) \( \text{CM}^Z A \cong (\text{mod}K)^{\oplus p + 1} \).

(d) \( \text{indCM}^Z A = \{ M, \Omega M, \cdots, \Omega^p M \} \) where the syzygies are taken as \( A \)-modules.

Example 14. Let \( \Lambda \) be a basic finite dimensional algebra of finite global dimension and \( e, f \in \Lambda \) idempotent elements. Then the algebra \( A = \Lambda \oplus (\Lambda e \otimes_K f \Lambda) \) is of finite global dimension if and only if \( f \Lambda e = 0 \). The algebra \( A \) is an IG algebra of infinite global dimension if and only if \( e = f \) and \( \dim \Lambda e = 1 \).

On the other hands, X-W. Chen [3] showed that \( \text{Sing} A \) is Hom-finite if and only if \( \dim f \Lambda e \leq 1 \). Thus we conclude that there are finite dimensional algebras \( A \) which is not IG but whose singular derived category \( \text{Sing} A \) is Hom-finite.

4.2. Classification. Using the categorical characterization of Theorem 8, we obtain the complete list of asid modules \( C \) when \( \Lambda \) is the path algebra of \( A_2 \)-quiver or \( A_3 \)-quiver in the following strategy.

Step 1. Classify admissible subcategories \( T \) of \( K^b(\text{proj}) \).

Step 2. For an admissible subcategory \( T \), classify bimodules \( C \) such that the functor \(- \otimes_K C \) acts on \( T \) as an equivalence and nilpotently acts on \( T^* \).

We give the list of \( T \) and \( C \) over \( \Lambda = K[1 \leftarrow 2] \). In the list, \( P_1, P_2 \) denote the indecomposable projective modules which correspond to the vertex 1, 2 respectively. \( I_2 \) is the indecomposable injective module which corresponds to the vertex 2. \( S_1^{\text{left}} \) denotes the simple left \( \Lambda \)-module which corresponds to the vertex 1. \( S_2^{\text{right}} \) denotes the simple right \( \Lambda \)-module which corresponds to the vertex 2.

(I) \( T = D^b(\text{mod}\Lambda) \) (precisely the case \( \alpha = 0 \)).

(II) \( T = \text{add}\{ P_1[i] \mid i \in \mathbb{Z} \} \)

(III) \( T = \text{add}\{ P_2[i] \mid i \in \mathbb{Z} \} \)

(IV) \( T = \text{add}\{ I_2[i] \mid i \in \mathbb{Z} \} \)

(V) \( T = 0 \) (precisely the case \( \text{gldimA} < \infty \)).

(V-1) \( C = (\Lambda e_2 \otimes_K e_1 \Lambda)^{\oplus n} \)

(V-2) \( C = (S_1^{\text{left}} \otimes_K e_2 \Lambda)^{\oplus n} \)

(V-3) \( C = (\Lambda e_1 \otimes_K S_2^{\text{right}})^{\oplus n} \)

For the list of \( A_3 \) case, we refer [7].

References


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