WHEN IS AN ABELIAN CATEGORY A QUANTUM PROJECTIVE SPACE?

IZURU MORI AND KENTA UEYAMA

Abstract. In noncommutative algebraic geometry, the noncommutative projective scheme associated to an AS-regular algebra is regarded as a quantum projective space. In this article, we will characterize a k-linear noetherian abelian category $\mathscr{C}$ such that $\mathscr{C}$ is equivalent to the noncommutative projective scheme associated to some right noetherian AS-regular algebra A over R.

1. Introduction

This article is based on our work [4].

In this section, we will explain the purpose of this article. Throughout, let k be a field.

Definition 1. (1) A (noetherian) algebraic triple consists of a k-linear (noetherian) category $\mathscr{C}$, an object $\mathcal{O} \in \mathscr{C}$, and a k-linear autoequivalence $s \in \operatorname{Aut}_k \mathscr{C}$. In this case, we also say that $(\mathcal{O}, s)$ is an algebraic pair for $\mathcal{C}$.

(2) A morphism of algebraic triples $(F, \theta, \mu) : (\mathscr{C}, \mathcal{O}, s) \to (\mathscr{C}', \mathcal{O}', s')$ consists of a k-linear functor $F : \mathscr{C} \to \mathscr{C}'$, an isomorphism $\theta : F(\mathcal{O}) \to \mathcal{O}'$ and a natural transformation $\mu : F \circ s \to s' \circ F$.

(3) Two algebraic triples $(\mathscr{C}, \mathcal{O}, s)$ and $(\mathscr{C}', \mathcal{O}', s')$ are isomorphic, denoted by $(\mathscr{C}, \mathcal{O}, s) \cong (\mathscr{C}', \mathcal{O}', s')$ if there exists a morphism of algebraic triples $(F, \theta, \mu) : (\mathscr{C}, \mathcal{O}, s) \to (\mathscr{C}', \mathcal{O}', s')$ such that F is an equivalence functor and μ is a natural isomorphism.

(4) For an algebraic triple $(\mathscr{C}, \mathcal{O}, s)$, we define an $\mathbb{N}$-graded k-algebra by

$$B(\mathscr{C}, \mathcal{O}, s)_{\geq 0} := \bigoplus_{i \in \mathbb{N}} \operatorname{Hom}_{\mathscr{C}}(\mathcal{O}, s^i \mathcal{O})$$

where the multiplication is given by the following rule: for $\alpha \in B(\mathscr{C}, \mathcal{O}, s)_i = \operatorname{Hom}_{\mathscr{C}}(\mathcal{O}, s^i \mathcal{O})$ and $\beta \in B(\mathscr{C}, \mathcal{O}, s)_j = \operatorname{Hom}_{\mathscr{C}}(\mathcal{O}, s^j \mathcal{O})$, we define $\alpha\beta := s^j(\alpha) \circ \beta \in \operatorname{Hom}_{\mathscr{C}}(\mathcal{O}, s^{i+j}\mathcal{O}) = B(\mathscr{C}, \mathcal{O}, s)_{i+j}$.

Example 2. Let X be a noetherian scheme over k, $\operatorname{coh} X$ the category of coherent sheaves on X, and $\mathcal{L} \in \operatorname{Pic} X$ an invertible sheaf on X. Then

$$(\operatorname{coh} X, \mathcal{O}_X, - \otimes_X \mathcal{L})$$

is a noetherian algebraic triple.

The detailed version of this paper has been submitted for publication elsewhere.

The first author was supported by JSPS Grant-in-Aid for Scientific Research (C) 16K05097 and JSPS Grant-in-Aid for Scientific Research (B) 16H03923. The second author was supported by JSPS Grant-in-Aid for Young Scientists (B) 15K17503.

Example 3. Let R be a right noetherian k-algebra, and $\operatorname{mod} R$ the category of finitely generated right R-modules. Then

$$\operatorname{Spec}_{\mathrm{nc}} R := (\operatorname{mod} R, R, \mathrm{id})$$

is a noetherian algebraic triple. One can check that $B(\operatorname{mod} R, R, \mathrm{id})_{\geq 0}$ is isomorphic to $R[x]$ where $\deg x = 1$. We say that $\operatorname{Spec}_{\mathrm{nc}} R$ is the noncommutative affine scheme associated to R since if R is commutative and $X = \operatorname{Spec} R$ is the affine scheme associated to R, then $\operatorname{Spec}_{\mathrm{nc}} R \cong (\operatorname{coh} X, \mathcal{O}_X, \mathrm{id})$.

Example 4. Let A be an $\mathbb{N}$-graded right noetherian k-algebra, and $\operatorname{grmod} A$ the category of finitely generated graded right A-modules. Then

$$\operatorname{GrSpec}_{\mathrm{nc}} A := (\operatorname{grmod} A, A, (1))$$

is a noetherian algebraic triple where (1) is the grade shift functor on $\operatorname{grmod} A$. One can check that $B(\operatorname{grmod} A, A, (1))_{\geq 0}$ is isomorphic to A.

Example 5. Let A be an $\mathbb{N}$-graded right noetherian k-algebra, and $\operatorname{fdim} A$ the full subcategory of $\operatorname{grmod} A$ consisting of finite dimensional modules. Then the Serre quotient category

$$\operatorname{tails} A := \operatorname{grmod} A / \operatorname{fdim} A$$

is a noetherian abelian category. We write $\pi : \operatorname{grmod} A \to \operatorname{tails} A$ for the quotient functor. Then we have a noetherian algebraic triple

$$\operatorname{Proj}_{\mathrm{nc}} A := (\operatorname{tails} A, \pi A, (1))$$

where (1) is the the grade shift functor on $\operatorname{tails} A$ induced by the grade shift functor on $\operatorname{grmod} A$. We say that $\operatorname{Proj}_{\mathrm{nc}} A$ is the noncommutative projective scheme associated to A since if A is commutative and generated in degree 1, and $X = \operatorname{Proj} A$ is the projective scheme associated to A, then $\operatorname{Proj}_{\mathrm{nc}} A \cong (\operatorname{coh} X, \mathcal{O}_X, - \otimes_X \mathcal{O}_X(1))$ by Serre's theorem [5]. (The category $\operatorname{tails} A$ itself is also often called the noncommutative projective scheme associated to A.)

By introducing the notion of ampleness for a noetherian algebraic triple, Artin and Zhang gave a criterion for a noetherian algebraic triple to be isomorphic to $\operatorname{Proj}_{\mathrm{nc}} A$ for some $\mathbb{N}$-graded right noetherian k-algebra A.

Definition 6 ([1, Section 4, p.250])**.** We say that an algebraic pair $(\mathcal{O}, s)$ for a k-linear noetherian abelian category $\mathscr{C}$ is ample if

(A1) for every $\mathcal{M} \in \mathscr{C}$, there exists a surjection $\bigoplus_{j=1}^{p} s^{-i_j}\mathcal{O} \to \mathcal{M}$ in $\mathscr{C}$ for some $i_1, \dots, i_p \geq 0$, and

(A2) for every surjection $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathscr{C}$, there exists $m \in \mathbb{Z}$ such that

$$\operatorname{Hom}_{\mathscr{C}}(s^{-i}\mathcal{O}, \phi) : \operatorname{Hom}_{\mathscr{C}}(s^{-i}\mathcal{O}, \mathcal{M}) \to \operatorname{Hom}_{\mathscr{C}}(s^{-i}\mathcal{O}, \mathcal{N})$$

is surjective for every $i \geq m$.

Roughly speaking, $(\mathcal{O}, s)$ is ample for $\mathscr{C}$ if and only if $\{s^{-i}\mathcal{O}\}_{i \geq m}$ is a set of progenerators for $\mathscr{C}$ for any $m \gg 0$. In fact, $(\mathcal{O}, \mathrm{id})$ is ample for $\mathscr{C}$ if and only if $\mathcal{O}$ is a progenerator for $\mathscr{C}$.

Theorem 7 ([1, Theorem 4.5])**.** *Let $(\mathscr{C}, \mathcal{O}, s)$ be a noetherian algebraic triple. Then $(\mathscr{C}, \mathcal{O}, s) \cong \operatorname{Proj_{nc}} A$ for some $\mathbb{N}$-graded right noetherian k-algebra A satisfying the condition "χ_1" if and only if*

(AZ1) $R := \operatorname{End}_{\mathscr{C}}(\mathcal{O})$ *is a right noetherian algebra,*
(AZ2) $\operatorname{Hom}_{\mathscr{C}}(\mathcal{M}, \mathcal{O}) \in \operatorname{mod} R$ *for every $\mathcal{M} \in \mathscr{C}$, and*
(AZ3) $(\mathcal{O}, s)$ *is ample for $\mathscr{C}$.*

If this is the case, then $A := B(\mathscr{C}, \mathcal{O}, s)_{\geq 0}$ is an $\mathbb{N}$-graded right noetherian k-algebra satisfying "χ_1" such that

$$(\mathscr{C}, \mathcal{O}, s) \cong \operatorname{Proj_{nc}} A.$$

The following example indicates that Artin-Zhang's theorem can be considered as a generalization of Morita theorem.

Example 8. If R is a right noetherian algebra, then the polynomial algebra $R[x]$ is an $\mathbb{N}$-graded right noetherian algebra satisfying "χ_1", and it is not difficult to show that $\operatorname{Proj_{nc}} R[x] \cong \operatorname{Spec_{nc}} R$ when $\deg x = 1$.

Let $\mathscr{C}$ be a k-linear noetherian abelian category. By Artin-Zhang's theorem, $\mathscr{C} \cong \operatorname{mod} R$ for some right noetherian algebra R if and only if there exists an object $\mathcal{O} \in \mathscr{C}$ such that $(\mathscr{C}, \mathcal{O}, \mathrm{id}) \cong \operatorname{Spec_{nc}} R \cong \operatorname{Proj_{nc}} R[x]$ for some right noetherian algebra R if and only if there exists an object $\mathcal{O} \in \mathscr{C}$ such that $(\mathcal{O}, \mathrm{id})$ satisfies the conditions (AZ1), (AZ2), (AZ3) if and only if there exists a progenerator $\mathcal{O}$ for $\mathscr{C}$.

We next recall the definition of an AS-regular algebra over R introduced in [2].

Definition 9 ([2, Definition 3.1])**.** A locally finite $\mathbb{N}$-graded right noetherian k-algebra A with $A_0 = R$ is called AS-regular over R of dimension d and of Gorenstein parameter ℓ if the following conditions are satisfied:

(1) $\operatorname{gldim} R < \infty$,
(2) $\operatorname{gldim} A = d < \infty$, and
(3) $\underline{\operatorname{Ext}}^i_A(R, A) \cong \begin{cases} 0 & (i \neq d) \\ DR(\ell) \text{ as graded right and left } A\text{-modules} & (i = d) \end{cases}$

where $\underline{\operatorname{Ext}}^i_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^i_{\operatorname{grmod} A}(M, N(n))$ and $D(-) := \operatorname{Hom}_k(-, k)$.

It is well-known that a commutative AS-regular algebra over k of dimension d is exactly a graded polynomial algebra $k[x_1, \ldots, x_d]$, so an AS-regular algebra is a noncommutative generalization of a polynomial algebra. The noncommutative projective scheme associated to an AS-regular algebra over R is called a quantum projective space over R. It is known that quantum projective spaces over k have nice properties as the projective space has, but their structures vary widely. To connect noncommutative algebraic geometry with other areas of mathematics, the following question is of major importance.

Question. When is a given noetherian algebraic triple isomorphic to a quantum projective space? That is, can we find a necessary and sufficient condition on a noetherian algebraic triple $(\mathscr{C}, \mathcal{O}, s)$ such that

$$(\mathscr{C}, \mathcal{O}, s) \cong \operatorname{Proj_{nc}} A \ \ (A\text{: right noetherian AS-regular over } R)?$$

By Theorem 7, to give an answer to this question, we should find extra conditions on an ample pair $(\mathcal{O}, s)$ for $\mathscr{C}$ such that $B(\mathscr{C}, \mathcal{O}, s)_{\geq 0}$ becomes AS-regular over R. The purpose of this article is to provide a complete answer to this question (Theorem 18).

2. Main Result

The canonical sheaf plays an essential role to study a projective scheme in commutative algebraic geometry. We first define a notion of canonical bimodule for an abelian category.

Definition 10. Let $\mathscr{C}$ be a Hom-finite k-linear category. A Serre functor for $\mathscr{C}$ is a k-linear autoequivalence $S \in \operatorname{Aut}_k \mathscr{C}$ such that there exists a bifunctorial isomorphism

$$F_{X,Y} \operatorname{Hom}_{\mathscr{C}}(X, Y) \to D \operatorname{Hom}_{\mathscr{C}}(Y, S(X))$$

for $X, Y \in \mathscr{C}$.

Definition 11. Let $\mathscr{C}$ be a Hom-finite k-linear abelian category. A bimodule $\mathcal{M}$ over $\mathscr{C}$ is an adjoint pair of functors from $\mathscr{C}$ to itself with the suggestive notation $\mathcal{M} = (- \otimes_{\mathscr{C}} \mathcal{M}, \mathcal{H}om_{\mathscr{C}}(\mathcal{M}, -))$. A bimodule $\mathcal{M}$ over $\mathscr{C}$ is invertible if $- \otimes_{\mathscr{C}} \mathcal{M}$ is an autoequivalence of $\mathscr{C}$. In this case, the inverse bimodule of $\mathcal{M}$ is defined by $\mathcal{M}^{-1} = (- \otimes_{\mathscr{C}} \mathcal{M}^{-1}, \mathcal{H}om_{\mathscr{C}}(\mathcal{M}^{-1}, -)) := (\mathcal{H}om_{\mathscr{C}}(\mathcal{M}, -), - \otimes_{\mathscr{C}} \mathcal{M})$.

Definition 12. Let $\mathscr{C}$ be a Hom-finite k-linear abelian category. A canonical bimodule for $\mathscr{C}$ is an invertible bimodule $\omega_{\mathscr{C}}$ over $\mathscr{C}$ such that, for some $n \in \mathbb{Z}$, the autoequivalence $- \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}[n]$ of $\mathcal{D}^b(\mathscr{C})$ induced by $- \otimes_{\mathscr{C}} \omega_{\mathscr{C}}$ is a Serre functor for $\mathcal{D}^b(\mathscr{C})$.

Remark 13. Let $\mathscr{C}$ be a Hom-finite k-linear abelian category.

1. Since the Serre functor for $\mathcal{D}^b(\mathscr{C})$ is unique, a canonical bimodule for $\mathscr{C}$ is unique if it exists.
2. If $\mathscr{C}$ has a canonical bimodule $\omega_{\mathscr{C}}$, and $- \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}[n]$ is the Serre functor for $\mathcal{D}^b(\mathscr{C})$, then it is easy to see that $\operatorname{gldim} \mathscr{C} = n < \infty$.

Example 14. If X is a noetherian smooth projective scheme over k, then the canonical sheaf ω_X on X is the canonical bimodule for $\operatorname{coh} X$.

Next we define a "relaxed" version of a helix.

Definition 15. Let $\mathcal{T}$ be a k-linear triangulated category.

1. A sequence of objects $\{E_0, \dots, E_{\ell-1}\}$ in $\mathcal{T}$ is called an exceptional sequence (resp. a relative exceptional sequence) if
   (a) $\operatorname{End}_{\mathcal{T}}(E_i) = k$ (resp. $\operatorname{gldim} \operatorname{End}_{\mathcal{T}} E_i < \infty$) for every $i = 0, \dots, \ell - 1$,
   (b) $\operatorname{Hom}_{\mathcal{T}}(E_i, E_i[q]) = 0$ for every $q \neq 0$ and every $i = 0, \dots, \ell - 1$, and
   (c) $\operatorname{Hom}_{\mathcal{T}}(E_i, E_j[q]) = 0$ for every q and every $0 \leq j < i \leq \ell - 1$.
2. A sequence of objects $\{E_0, \dots, E_{\ell-1}\}$ in $\mathcal{T}$ is called full if the thick subcategory of $\mathcal{T}$ containing $\{E_0, \dots, E_{\ell-1}\}$ equals $\mathcal{T}$.

Definition 16. Let $\mathscr{C}$ be a Hom-finite k-linear abelian category having the canonical bimodule $\omega_{\mathscr{C}}$.

1. A sequence of objects $\{E_i\}_{i \in \mathbb{Z}}$ in $\mathcal{D}^b(\mathscr{C})$ is called a (relative) helix of period ℓ if, for each $i \in \mathbb{Z}$,

(a) $\{E_i, \dots, E_{i+\ell-1}\}$ is a (relative) exceptional sequence for $\mathcal{D}^b(\mathscr{C})$, and
(b) $E_{i+\ell} \cong E_i \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}^{-1}$.

(2) A relative helix $\{E_i\}_{i\in\mathbb{Z}}$ of period ℓ is called full if $\{E_i, \dots, E_{i+\ell-1}\}$ is full for every $i \in \mathbb{Z}$.

(3) A relative helix $\{E_i\}_{i\in\mathbb{Z}}$ of period ℓ is called geometric if $\mathrm{Hom}_{\mathscr{C}}(E_i, E_j[q]) = 0$ for every $q \neq 0$ and every $i \le j$.

Example 17. If $X = \mathbb{P}^{d-1}$, then $\{\mathcal{O}_X(i)\}_{i\in\mathbb{Z}}$ is a full geometric helix of period d for $\mathcal{D}^b(\operatorname{coh} X)$.

We can now state our main theorem.

Theorem 18. *A noetherian algebraic triple $(\mathscr{C}, \mathcal{O}, s)$ is isomorphic to $\mathrm{Proj}_{\mathrm{nc}} A$ for some right noetherian AS-regular algebra A over R of Gorenstein parameter ℓ if and only if*

(AS1) *$\mathscr{C}$ is a* Hom*-finite k-linear abelian category having the canonical bimodule $\omega_{\mathscr{C}}$,*
(AS2) *$(\mathcal{O}, s)$ is ample for $\mathscr{C}$, and*
(AS3) *$\{s^i\mathcal{O}\}_{i\in\mathbb{Z}}$ is a full geometric relative helix of period ℓ for $\mathcal{D}^b(\mathscr{C})$.*

In fact, if (AS1), (AS2), and (AS3) are satisfied, then $A = B(\mathscr{C}, \mathcal{O}, s)_{\geq 0}$ is a right noetherian AS-regular algebra over $R = \mathrm{End}_{\mathscr{C}}(\mathcal{O})$ of dimension $\operatorname{gldim} \mathscr{C} + 1$ *and of Gorenstein parameter ℓ such that $(\mathscr{C}, \mathcal{O}, s) \cong \mathrm{Proj}_{\mathrm{nc}} A$.*

Example 19. Let $A = k[x_1, \dots, x_d]$ be a polynomial algebra such that $\operatorname{char} k = 0$, $\deg x_i = 1$, and $d \geq 2$. Let G be a finite subgroup of $\mathrm{SL}_d(k)$ acting linearly on A, and A^G the fixed subring of A. Assume that A^G is an isolated singularity. Then

- $\pi A^G(-d)$ is the canonical bimodule for $\operatorname{tails} A^G$,
- $(\pi A, (1))$ is ample for $\operatorname{tails} A^G$, and
- $\{\pi A(i)\}_{i\in\mathbb{Z}}$ is a full geometric relative helix of period d for $\mathcal{D}^b(\operatorname{tails} A^G)$.

Hence we have

$$(\operatorname{tails} A^G, \pi A, (1)) \cong \mathrm{Proj}_{\mathrm{nc}} S$$

where $S := B(\operatorname{tails} A^G, \pi A, (1))_{\geq 0} \cong A * G$ is a right noetherian AS-regular algebra over kG of dimension d and of Gorenstein parameter d (see [3]).

Example 20. Let $X = \mathbb{P}^1 \times \mathbb{P}^1$. There is no ample invertible sheaf $\mathcal{L} \in \operatorname{Pic} X$ such that $\{\mathcal{L}^{\otimes i}\}_{i\in\mathbb{Z}}$ is a full geometric helix for $\mathcal{D}^b(\operatorname{coh} X)$. However, take an automorphism $\sigma \in \mathrm{Aut}_k X$ of order 2 which switches the components of X, and put $s := \sigma^*(- \otimes_X \mathcal{O}_X(0,1)) \in \mathrm{Aut}_k(\operatorname{coh} X)$. Then

- $\mathcal{O}_X(-2,-2)$ is the canonical bimodule for $\operatorname{coh} X$,
- $(\mathcal{O}_X, s)$ is ample for $\operatorname{coh} X$, and
- $\{s^i\mathcal{O}_X\}_{i\in\mathbb{Z}}$ is a full geometric helix of period 4 for $\mathcal{D}^b(\operatorname{coh} X)$.

Hence we have

$$(\operatorname{coh} X, \mathcal{O}_X, s) \cong \mathrm{Proj}_{\mathrm{nc}} S$$

where $S := B(\operatorname{coh} X, \mathcal{O}_X, s)_{\geq 0} \cong k\langle x, y\rangle/(x^2y - yx^2, y^2x - xy^2)$ is a right noetherian AS-regular algebra over k of dimension 3 and of Gorenstein parameter 4.

References

[1] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. Math.* **109** (1994), no. 2, 228–287.

[2] H. Minamoto and I. Mori, The structure of AS-Gorenstein algebras, *Adv. Math.* **226** (2011), no. 5, 4061–4095.

[3] I. Mori and K. Ueyama, Ample group actions on AS-regular algebras and noncommutative graded isolated singularities, *Trans. Amer. Math. Soc.* **368** (2016), no. 10, 7359–7383.

[4] I. Mori and K. Ueyama, A categorical characterization of quantum projective spaces, preprint, `arXiv:1708.00167`.

[5] J.-P. Serre, Faisceaux algébriques cohérents, *Ann. of Math. (2)* **61** (1955), 197–278.

Izuru Mori:
Department of Mathematics
Faculty of Science
Shizuoka University
Shizuoka 422-8529 JAPAN

E-mail address: `mori.izuru@shizuoka.ac.jp`

Kenta Ueyama:
Department of Mathematics
Faculty of Education
Hirosaki University
Hirosaki 036-8560 JAPAN

E-mail address: `k-ueyama@hirosaki-u.ac.jp`