THE MODULI OF SUBALGEBRAS OF THE FULL MATRIX RING OF DEGREE 3

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Abstract. There exist 26 equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ for any algebraically closed field k. We introduce the moduli of subalgebras of the full matrix ring of degree 3, in other words, the moduli of molds. We describe the moduli of rank d molds of degree 3 for $d = 2, 3$.

Key Words: Subalgebra, Matrix ring, Mold, Moduli of molds.

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1. Introduction

In this paper, we discuss the moduli of subalgebras of the full matrix ring of degree 3. For investigating R-subalgebras A of $\mathrm{M}_n(R)$ over a commutative ring R, or subsheaves $\mathcal{A}$ of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ on a scheme X, it is a good way to investigate the moduli of subalgebras of the full matrix ring, in other words, the moduli of molds. For constructing the moduli, we need to introduce the notion of molds. A mold $\mathcal{A} \subset \mathrm{M}_n(\mathcal{O}_X)$ of degree n on a scheme X is a subsheaf of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ such that $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves. This definition allows us to construct the moduli $\mathrm{Mold}_{n,d}$ of rank d molds of degree n as a closed subscheme of the Grassmann scheme $\mathrm{Grass}(d, n^2)$. By the description of the moduli of molds of degree 3, we obtain rich results on R-subalgebras of $\mathrm{M}_3(R)$ over arbitrary commutative ring R. Furthermore, we can obtain a hint to construct the moduli of representations for each mold of degree 3 as in the degree 2 case ([5]).

Another reason why we investigate the moduli of molds is that it is closely related to the moduli of algebras in the sense of Gabriel ([1]). There exists a canonical morphism from the moduli of algebras to the moduli of molds. Moreover, we can deal with the moduli of molds functorially, since it represents a certain contravariant functor and there is a universal family of subalgebras on the moduli of molds. The relation between the moduli of molds and the moduli of algebras will be discussed in another paper.

We begin with the following definition.

Definition 1. Let k be an algebraically closed field. Let A, B be k-subalgebras of $\mathrm{M}_n(k)$. We say that A and B are *equivalent* if there exists $P \in \mathrm{GL}_n(k)$ such that $P^{-1}AP = B$.

Let us classify the equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k.

Theorem 2 ([6]). *There exist* 26 *equivalence classes of k-subalgebras of* $\mathrm{M}_3(k)$ *for any algebraically closed field k.*

The detailed version of this paper will be submitted for publication elsewhere.

(1) $\mathrm{M}_3(k)$

(2) $\mathrm{P}_{2,1}(k) := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(3) $\mathrm{P}_{1,2}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(4) $\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(5) $\mathrm{C}_3(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a \in k \right\}$

(6) $\mathrm{D}_3(k) := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(7) $(\mathrm{C}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b \in k \right\}$

(8) $(\mathrm{N}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & c & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(9) $(\mathrm{B}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} * & * & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(10) $(\mathrm{M}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(11) $\mathrm{J}_3(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(12) $\mathrm{N}_3(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in k \right\}$

(13) $S_1(k) := \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}$

(14) $S_2(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(15) $S_3(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(16) $S_4(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(17) $S_5(k) := \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(18) $S_6(k) := \left\{ \begin{pmatrix} a & c & d \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in k \right\}$

(19) $S_7(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & a & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in k \right\}$

(20) $S_8(k) := \left\{ \begin{pmatrix} a & c & d \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in k \right\}$

(21) $S_9(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & d \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d \in k \right\}$

(22) $S_{10}(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & e \end{pmatrix} \middle| \, a, b, c, d, e \in k \right\}$

(23) $S_{11}(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d, e \in k \right\}$

(24) $S_{12}(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & e & d \\ 0 & 0 & e \end{pmatrix} \middle| \, a, b, c, d, e \in k \right\}$

(25) $S_{13}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(26) $S_{14}(k) := \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

For a long proof of Theorem 2, see [6]. It is interesting to investigate how 26 types of subalgebras are contained in the moduli $\mathrm{Mold}_{3,d}$ of molds ($d = 1, 2, \ldots, 9$).

The organization of this paper is as follows. In Section 2, we define the moduli of molds. We describe $\mathrm{Mold}_{3,d}$ for $d = 1, 6, 7, 8, 9$. In Section 3, we describe $\mathrm{Mold}_{3,2}$. In Section 4, we describe $\mathrm{Mold}_{3,3}$. In Section 5, we deal with the degree 2 case as an appendix.

2. Preliminaries

In this section, we define the moduli of subalgebras of the full matrix ring, in other words, the moduli of molds. We describe the moduli of rank d molds of degree 3 for $d = 1, 6, 7, 8, 9$.

For introducing the moduli of subalgebras of the full matrix ring, we define molds on schemes.

Definition 3 ([4, Definition 1.1])**.** Let X be a scheme. A subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is said to be a *mold* of degree n on X if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on X. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf on X. For a commutative ring R, we say that an R-subalgebra $A \subseteq \mathrm{M}_n(R)$ is a *mold* of degree n over R if A is a mold of degree n on $\mathrm{Spec}R$.

Remark 4. A mold $A \subset \mathrm{M}_n(R)$ over a commutative ring R is an R-subalgebra A of $\mathrm{M}_n(R)$ satisfying that A and $\mathrm{M}_n(R)/A$ are projective R-modules. In particular, $A_\wp$ is a free $R_\wp$-module for $\wp \in \mathrm{Spec}R$. We assume that $\mathrm{rank}_{R_\wp} A_\wp$ is constant for $\wp \in \mathrm{Spec}R$. Then $A \subset \mathrm{M}_n(R)$ determines an R-valued point of $\mathrm{Grass}(d, n^2)$, where $d = \mathrm{rank}A$. Here the Grassmann scheme $\mathrm{Grass}(m, n)$ is the scheme over $\mathbb{Z}$ parameterizing rank m subbundles of the trivial rank n vector bundle (For example, see [2, Lecture 5]).

We define the moduli of molds.

Proposition 5 ([4, Definition and Proposition 1.1])**.** *The following contravariant functor is representable by a $\mathbb{Z}$-scheme* $\mathrm{Mold}_{n,d}$.

$$\begin{array}{rcccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \,\middle|\, \textit{rank } d \textit{ mold of degree } n \textit{ on } X \right\} \end{array}$$

Moreover, $\mathrm{Mold}_{n,d}$ *is a closed subscheme of the Grassmann scheme* $\mathrm{Grass}(d, n^2)$.

Proof. Let E be the universal rank d subbundle of the trivial rank n^2 vector bundle on $\mathrm{Grass}(d, n^2)$. We regard the trivial rank n^2 vector bundle as $\mathrm{M}_n(\mathcal{O}_{\mathrm{Grass}(d,n^2)})$. Let $p \in \mathrm{Grass}(d, n^2)$. Let $A_1, \dots, A_{n^2}$ be an $\mathcal{O}_{U_p}$-basis of $\mathrm{M}_n(\mathcal{O}_{U_p})$ such that $A_1, \dots, A_d$ is an $\mathcal{O}_{U_p}$-basis of $E\,|_{U_p}$ on a neighborhood U_p of p. We can write $I_n = \sum_{k=1}^{n^2} a_k A_k$ and $A_i A_j = \sum_{k=1}^{n^2} c_{ij}^k A_k$ for $i, j = 1, 2, \dots d$, where $a_k, c_{ij}^k \in \mathcal{O}_{\mathrm{Grass}(d,n^2)}(U_p)$. The condition $a_k = 0$ for $k = d+1, \dots, n^2$ and that $c_{ij}^k = 0$ for $1 \le i, j \le d$ and $k = d+1, \dots, n^2$ defines a closed subscheme of U_p. By gluing such closed subschemes of U_p for each $p \in \mathrm{Grass}(d, n^2)$, we can obtain a closed subscheme $\mathrm{Mold}_{n,d}$ of $\mathrm{Grass}(d, n^2)$. It is easy to check that $\mathrm{Mold}_{n,d}$ represents the contravariant functor in the statement. □

Remark 6. There exists the universal mold $\mathcal{A}_{n,d} \subset \mathrm{M}_n(\mathcal{O}_{\mathrm{Mold}_{n,d}})$ on the moduli $\mathrm{Mold}_{n,d}$. Giving a rank d mold $A \subset \mathrm{M}_n(R)$ over a commutative ring R is equivalent to giving a morphism $\mathrm{Spec}R \to \mathrm{Mold}_{n,d}$. By the morphism $\phi : \mathrm{Spec}R \to \mathrm{Mold}_{n,d}$ corresponding to $A \subset \mathrm{M}_n(R)$, we obtain $A = \phi^*(\mathcal{A}_{n,d}) := \mathcal{A}_{n,d} \otimes_{\mathcal{O}_{\mathrm{Mold}_{n,d}}} R$.

Example 7 ([6])**.** Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned}
\mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\
\mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

Let us explain Example 7. When $d = 1$, $A = RI_3 \subset \mathrm{M}_3(R)$ is the unique rank 1 mold over a commutative ring R. Then $\mathrm{Mold}_{3,1}$ is isomorphic to $\mathrm{Spec}\mathbb{Z}$, which is the final object in the category of schemes. When $d = 9$, $A = \mathrm{M}_3(R)$ is the unique rank 9 mold over a commutative ring R. Then $\mathrm{Mold}_{3,9}$ is also isomorphic to $\mathrm{Spec}\mathbb{Z}$. When $d = 8$, there exists no rank 8 mold $A \subset \mathrm{M}_3(R)$ over any commutative ring R. Hence $\mathrm{Mold}_{3,8} = \emptyset$.

When $d = 6$, the set of k-rational points of $\mathrm{Mold}_{3,6} = \mathrm{Flag}$ coincides with $\{P\mathrm{B}_3(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\}$ for a field k, where

$$\mathrm{B}_3(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}.$$

When $d = 7$, $\mathrm{Mold}_{3,7} = \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}$.
Let $\mathrm{P}_{2,1}(k) := \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$ and $\mathrm{P}_{1,2}(k) := \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$.
The set of k-rational points of $\mathrm{Mold}_{3,7} = \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}$ coincides with

$$\{P\mathrm{P}_{2,1}(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\} \coprod \{P\mathrm{P}_{1,2}(k)P^{-1} \mid P \in \mathrm{GL}_3(k)\},$$

where k is a field.

In the following sections, we discuss $\mathrm{Mold}_{3,2}$ and $\mathrm{Mold}_{3,3}$.

3. The moduli $\mathrm{Mold}_{3,2}$

In this section, we deal with $\mathrm{Mold}_{3,2}$. Let k be an algebraically closed field. There exist two equivalence classes of 2-dimensional k-subalgebras of $\mathrm{M}_3(k)$:

$$(\mathrm{C}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \; a, b \in k \right\}$$

and

$$S_1(k) := \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \; a, b \in k \right\}.$$

We can classify 3×3-matrices into three types: Regular matrices, subregular matrices, and scalar matrices.

Definition 8 ([6])**.** Let M_3 be the scheme of 3×3-matrices over $\mathbb{Z}$. In other words, $\mathrm{M}_3 \cong \mathbb{A}^9_{\mathbb{Z}}$ and we can consider the universal matrix A on M_3. We define the open subscheme $\mathrm{M}_3^{\mathrm{reg}}$ consisting of non-derogatory matrices (or regular matrices) by

$$\mathrm{M}_3^{\mathrm{reg}} := \{x \in \mathrm{M}_3 \mid I_3, A, A^2 \text{ are linearly independent in } \mathrm{M}_3(k(x))\},$$

where $k(x)$ is the residue field of x. We denote by $\mathrm{M}_3^{\mathrm{scalar}}$ the closed subschemes consisting of scalar matrices. We also define the subscheme $\mathrm{M}_3^{\mathrm{sr}}$ of M_3 by

$$\mathrm{M}_3^{\mathrm{sr}} := \left\{ x \in \mathrm{M}_3 \;\middle|\; \begin{array}{c} I_3, A \text{ are linearly independent in } \mathrm{M}_3(k(x)) \text{ and } A^2 = c_1 A + c_0 I_3 \\ \text{on a neighborhood } U_x \text{ of } x \text{ for some } c_1, c_0 \in \mathcal{O}_{\mathrm{M}_3}(U_x) \end{array} \right\}.$$

Then M_3 can be divided into the following three subschemes:

$$\mathrm{M}_3 = \mathrm{M}_3^{\mathrm{reg}} \coprod \mathrm{M}_3^{\mathrm{sr}} \coprod \mathrm{M}_3^{\mathrm{scalar}}.$$

Roughly speaking, if the degree of the minimal polynomial for a 3×3-matrix A is 3, 2, or 1, then we call A regular, subregular, or scalar, respectively. We denote by $\mathrm{M}_3^{\mathrm{sr}}(R)$ the set of subregular matrices of $\mathrm{M}_3(R)$ over a commutative ring R. (This is compatible with the notation of the set of R-valued points of the scheme $\mathrm{M}_3^{\mathrm{sr}}$.)

For describing $\mathrm{Mold}_{3,2}$, we deal with subregular matrices.

Proposition 9 ([6])**.** *Let R be a local ring. For $A \in \mathrm{M}_3^{\mathrm{sr}}(R)$, there exists $P \in \mathrm{GL}_3(R)$ such that*

$$P^{-1}AP = \begin{pmatrix} a & 1 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix}.$$

Moreover, $a, b \in R$ are determined by only A (not by P).

Remark 10. Proposition 9 implies that $A \in \mathrm{M}_3^{\mathrm{sr}}(R)$ has eigenvalues a, b, b over arbitrary commutative ring R. In particular, even if a field k is not algebraically closed, then $A \in \mathrm{M}_3^{\mathrm{sr}}(k)$ has eigenvalues a, b, b over k.

Definition 11 ([6])**.** We call $a, b \in R$ in Proposition 9 the *a-invariant* and the *b-invariant* of A, respectively.

The subscheme $\mathrm{M}_3^{\mathrm{sr}}$ is the moduli of 3×3 subregular matrices. For the universal subregular matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, we can define the a-invariant and the b-invariant of A on $\mathrm{M}_3^{\mathrm{sr}}$.

Definition 12 ([6])**.** We denote by $a(A), b(A) \in \mathcal{O}_{\mathrm{M}_3^{\mathrm{sr}}}(\mathrm{M}_3^{\mathrm{sr}})$ the a-invariant and b-invariant of the universal matrix A on $\mathrm{M}_3^{\mathrm{sr}}$, respectively. These are PGL_3-invariant, where the group scheme PGL_3 acts on $\mathrm{M}_3^{\mathrm{sr}}$ by $A \mapsto P^{-1}AP$.

Here we introduce (universal) geometric quotients ([3]).

Definition 13 ([3, Definitions 0.6 and 0.7])**.** Let X be a scheme over a scheme S. Let G be a group scheme over S. For a given group action $\sigma : G \times_S X \to X$ over S, a pair (Y, ϕ) consisting of a scheme Y over S and an S-morphism $\phi : X \to Y$ is called a *geometric quotient* of X by G if

(1) $\phi \circ \sigma = \phi \circ p_2$, where $p_2 : G \times_S X \to X$ is the second projection.

(2) ϕ is surjective, and the image of $G \times_S X \stackrel{(\sigma,p_2)}{\to} X \times_S X$ coincides with $X \times_Y X$.
(3) ϕ is submersive. In other words, $U \subset Y$ is open if and only if $\phi^{-1}(U) \subset X$ is open.
(4) $\mathcal{O}_Y = \phi_*(\mathcal{O}_X)^G$, where $\phi_*(\mathcal{O}_X)^G$ is the subsheaf of $\phi_*(\mathcal{O}_X)$ consisting of G-invariant functions. In other words, for an open set U of Y and for $f \in \phi_*(\mathcal{O}_X)(U)$, $f \in \phi_*(\mathcal{O}_X)^G(U)$ if and only if f induces the following commutative diagram:

$$\begin{array}{ccc} G \times \phi^{-1}(U) & \stackrel{\sigma}{\to} & \phi^{-1}(U) \\ p_2 \downarrow & & \downarrow F \\ \phi^{-1}(U) & \stackrel{F}{\to} & \mathbb{A}^1_S. \end{array}$$

Here F is the morphism defined by f.

We say that (Y, ϕ) is a *universal geometric quotient* of X by G if (Y', ϕ') is a geometric quotient for any S-morphism $Y' \to Y$, where $\phi' : X' := X \times_S Y' \to Y'$ is induced by ϕ.

We also introduce the following definition.

Definition 14 ([7]). Let $f : X \to S$ be a morphism of schemes. Assume that f is locally of finite type. We say that f is of *relative dimension* d if all non-empty fibers $f^{-1}(s)$ of f are equidimensional of dimension d.

Proposition 15 ([6]). *Let $\pi : \mathrm{M}_3^{\mathrm{sr}} \to \mathbb{A}^2_{\mathbb{Z}}$ be the morphism defined by $A \mapsto (a(A), b(A))$. Then π gives a universal geometric quotient by PGL_3. Moreover, $\mathrm{M}_3^{\mathrm{sr}}$ is a smooth integral scheme of relative dimension 6 over $\mathbb{Z}$.*

The scheme $\mathrm{Mold}_{3,2}$ is the moduli of 2-dimensional subalgebras of M_3. Let $\langle A \rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{sr}}$. We define $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$ by $A \mapsto \langle A \rangle$.

Proposition 16 ([6]). *The morphism $\phi : \mathrm{M}_3^{\mathrm{sr}} \to \mathrm{Mold}_{3,2}$ is smooth and surjective.*

Let us describe $\mathrm{Mold}_{3,2}$ explicitly. Let $V := \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a rank 3 trivial vector bundle on $\mathrm{Spec}\mathbb{Z}$. We denote by $\mathbb{P}_*(V)$ the projective plane consisting of subline bundles of V. We also denote by $\mathbb{P}^*(V)$ the projective plane consisting of rank 2 subbundles of V.

Let us define a morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ as follows. Let X be a scheme, and let (L, W) be an X-valued point of $\mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a rank 1 subbundle and a rank 2 subbundle of $V \otimes_{\mathbb{Z}} \mathcal{O}_X$, respectively. Set $V_X := V \otimes_{\mathbb{Z}} \mathcal{O}_X$. We can regard $f \in \mathcal{H}om_{\mathcal{O}_X}(V_X/W, L)$ as an element of $\mathcal{E}nd_{\mathcal{O}_X}(V_X)$ by

$$V_X \stackrel{\text{proj.}}{\to} V_X/W \stackrel{f}{\to} L \hookrightarrow V_X.$$

We denote by $\xi(L, W)$ the subsheaf of $\mathcal{O}_X$-algebras of $\mathrm{End}_{\mathcal{O}_X}(V_X)$ generated by $\{f \in \mathrm{Hom}_{\mathcal{O}_X}(V_X/W, L)\}$ and id_{V_X}. Since rank $\mathcal{H}om_{\mathcal{O}_X}(V_X/W, L) = 1$, we see that $\xi(L, W)$ is a rank 2 mold on X. We define a morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ by $(L, W) \mapsto \xi(L, W)$.

Theorem 17 ([6]). *The morphism $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ is an isomorphism.*

Remark 18. For simplicity, let us consider the discussion above over a field k. Let V be a 3-dimensional vector space over k. Let $(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V)$. In other words, let L and W be a 1-dimensional subspace and a 2-dimensional subspace of V, respectively. We can regard $f \in \mathrm{Hom}_k(V/W, L)$ as an element of $\mathrm{End}_k(V)$ by

$$V \overset{\text{proj.}}{\to} V/W \overset{f}{\to} L \hookrightarrow V.$$

We denote by $\xi(L, W)$ the k-subalgebra of $\mathrm{End}_k(V)$ generated by $\{f \in \mathrm{Hom}_k(V/W, L)\}$ and id_V. Since $\dim \mathrm{Hom}_k(V/W, L) = 1$, we see that $\xi(L, W)$ is a 2-dimensional k-subalgebra. Theorem 17 shows that any 2-dimensional k-subalgebra A of $\mathrm{M}_3(k)$ can be obtained as $\xi(L, W)$ for some (L, W). Such (L, W) is uniquely determined by A.

Recall the following two types of 2-dimensional k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k:

$$(\mathrm{C}_2 \times \mathrm{D}_1)(k) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b \in k \right\},$$

$$S_1(k) = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}.$$

Definition 19 ([6])**.** We define an open subscheme $\mathrm{M}_3^{\mathrm{C}_2 \times \mathrm{D}_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{\mathrm{C}_2 \times \mathrm{D}_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) \neq 0\}.$$

We also define a closed subscheme $\mathrm{M}_3^{S_1}$ of $\mathrm{M}_3^{\mathrm{sr}}$ by

$$\mathrm{M}_3^{S_1} := \{A \in \mathrm{M}_3^{\mathrm{sr}} \mid a(A) - b(A) = 0\}.$$

Similarly, we define subschemes $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ of $\mathrm{Mold}_{3,2}$. Geometric points of $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}$ and $\mathrm{Mold}_{3,2}^{S_1}$ correspond to subalgebras which are equivalent to $\mathrm{C}_2 \times \mathrm{D}_1$ and S_1, respectively.

Set $\mathrm{Flag} := \{(L, W) \in \mathbb{P}_*(V) \times \mathbb{P}^*(V) \mid L \subset W\} \subset \mathbb{P}_*(V) \times \mathbb{P}^*(V)$.

Theorem 20 ([6])**.** *The isomorphism* $\xi : \mathbb{P}_*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,2}$ *induces* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1} \cong \mathbb{P}_*(V) \times \mathbb{P}^*(V) \setminus \mathrm{Flag}$ *and* $\mathrm{Mold}_{3,2}^{S_1} \cong \mathrm{Flag}$*. In particular,* $\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}$ *is a smooth integral scheme of relative dimension* 4 *over* $\mathbb{Z}$*, and* $\mathrm{Mold}_{3,2}^{S_1}$ *is a smooth integral scheme of relative dimension* 3 *over* $\mathbb{Z}$*.*

Corollary 21 ([6])**.** *For the finite field* $\mathbb{F}_q$*,*

$$\sharp\{A \mid 2\text{-dimensional subalgebra of } \mathrm{M}_3(\mathbb{F}_q)\} = \sharp\left(\mathbb{P}^2(\mathbb{F}_q) \times \mathbb{P}^2(\mathbb{F}_q)\right) = (q^2 + q + 1)^2.$$

Moreover,

$$\sharp\mathrm{Mold}_{3,2}^{\mathrm{C}_2 \times \mathrm{D}_1}(\mathbb{F}_q) = q^2(q^2 + q + 1),$$
$$\sharp\mathrm{Mold}_{3,2}^{\mathrm{S}_1}(\mathbb{F}_q) = (q^2 + q + 1)(q + 1).$$

4. The moduli $\mathrm{Mold}_{3,3}$

In this section, we deal with $\mathrm{Mold}_{3,3}$. There are seven types of 3-dimensional k-subalgebras of $\mathrm{M}_3(k)$ over an algebraically closed field k:

(1) $\mathrm{D}_3(k) := \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{M}_3(k) \right\}$

(2) $(\mathrm{N}_2 \times \mathrm{D}_1)(k) := \left\{ \begin{pmatrix} a & c & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(3) $\mathrm{J}_3(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(4) $S_2(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(5) $S_3(k) := \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(6) $S_4(k) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

(7) $S_5(k) := \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in k \right\}$

We define the regular part $\mathrm{M}_n^{\mathrm{reg}}$ of M_n.

Definition 22 ([6]). Let M_n be the scheme of $n \times n$-matrices over $\mathbb{Z}$. The scheme M_n is isomorphic to the affine space $\mathbb{A}_{\mathbb{Z}}^{n^2}$. Let A be the universal matrix on M_n. The open subscheme $\mathrm{M}_n^{\mathrm{reg}}$ of M_n is defined by

$$\mathrm{M}_n^{\mathrm{reg}} := \{x \in \mathrm{M}_n \mid I_n, A, A^2, \ldots, A^{n-1} : \text{linearly independent in } \mathrm{M}_n(k(x))\},$$

where $k(x)$ is the residue field of x. We call $\mathrm{M}_n^{\mathrm{reg}}$ the *regular part* of M_n. For a commutative ring R, we call a matrix $A \in \mathrm{M}_n^{\mathrm{reg}}(R)$ *regular* or *non-derogatory.*

Proposition 23 (*cf.* [6]). *Let R be a local ring. Let $A \in \mathrm{M}_n^{\mathrm{reg}}(R)$. There exists $P \in \mathrm{GL}_n(R)$ such that*

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -c_n \\ 1 & 0 & 0 & \ddots & 0 & -c_{n-1} \\ 0 & 1 & 0 & \ddots & 0 & -c_{n-2} \\ 0 & 0 & 1 & \ddots & 0 & -c_{n-3} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_1 \end{pmatrix}.$$

Note that $x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ is the characteristic polynomial of A.

The group scheme PGL_n acts on $\mathrm{M}_n^{\mathrm{reg}}$ by $A \mapsto P^{-1}AP$. We define $\pi : \mathrm{M}_n^{\mathrm{reg}} \to \mathbb{A}_{\mathbb{Z}}^n$ by $A \mapsto (c_1, c_2, \ldots, c_n)$, where $x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$ is the characteristic polynomial of A.

Theorem 24 ([6])**.** *The morphism $\pi : \mathrm{M}_n^{\mathrm{reg}} \to \mathbb{A}_{\mathbb{Z}}^n$ is a universal geometric quotient by PGL_n.*

Let us consider $\mathrm{Mold}_{3,3}$. Let $\langle A\rangle$ be the subalgebra generated by A for $A \in \mathrm{M}_3^{\mathrm{reg}}$. We define $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ by $A \mapsto \langle A\rangle$.

Proposition 25 ([6])**.** *The morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}$ is smooth and surjective.*

Definition 26 ([6])**.** We define an open subscheme $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$ of $\mathrm{Mold}_{3,3}$ by $\mathrm{Mold}_{3,3}^{\mathrm{reg}} := \psi(\mathrm{M}_3^{\mathrm{reg}})$.

Remark 27. Let k be a field. For a k-rational point A of $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$, there exists $X \in \mathrm{M}_3(k)$ such that $A = kI_3 + kX + kX^2$ except for the case that $k = \mathbb{F}_2$ and $A = P^{-1}\mathrm{D}_3(\mathbb{F}_2)P$ for some $P \in \mathrm{GL}_3(\mathbb{F}_2)$. Here $\mathrm{D}_3(k)$ is the k-subalgebra of $\mathrm{M}_3(k)$ consisting of diagonal matrices. The $\mathbb{F}_2$-rational point $A = P^{-1}\mathrm{D}_3(\mathbb{F}_2)P$ is also contained in $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$, since $A \otimes_{\mathbb{F}_2} \mathbb{F}_4 = P^{-1}\mathrm{D}_3(\mathbb{F}_4)P$ can be generated by some $X \in \mathrm{M}_3^{\mathrm{reg}}(\mathbb{F}_4)$.

The scheme $\mathrm{M}_3^{\mathrm{reg}}$ has the following stratification of subschemes.

Theorem 28 ([6])**.** *The smooth integral scheme $\mathrm{M}_3^{\mathrm{reg}}$ over $\mathbb{Z}$ has a stratification of subschemes*

$$\mathrm{M}_3^{\mathrm{reg}} = \mathrm{M}_3^{\mathrm{D}_3} \coprod \mathrm{M}_3^{\mathrm{N}_2\times\mathrm{D}_1} \coprod \mathrm{M}_3^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2} \coprod \mathrm{M}_3^{\mathrm{J}_3} \coprod \mathrm{M}_3^{\mathrm{J}_3/\mathbb{F}_3},$$

which have the following properties:

(1) $\mathrm{M}_3^{\mathrm{D}_3}$ *is a smooth integral scheme of relative dimension* 9 *over* $\mathbb{Z}$.
(2) $\mathrm{M}_3^{\mathrm{N}_2\times\mathrm{D}_1}$ *is a smooth integral scheme of relative dimension* 8 *over* $\mathbb{Z}[1/2]$.
(3) $\mathrm{M}_3^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2}$ *is a smooth variety of dimension* 8 *over* $\mathbb{F}_2$.
(4) $\mathrm{M}_3^{\mathrm{J}_3}$ *is a smooth integral scheme of relative dimension* 7 *over* $\mathbb{Z}[1/3]$.
(5) $\mathrm{M}_3^{\mathrm{J}_3/\mathbb{F}_3}$ *is a smooth variety of dimension* 7 *over* $\mathbb{F}_3$.

The scheme $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$ also has the following stratification of subschemes.

Theorem 29 ([6])**.** *The smooth integral scheme $\mathrm{Mold}_{3,3}^{\mathrm{reg}}$ of relative dimension 6 over $\mathbb{Z}$ has a stratification of subschemes*

$$\mathrm{Mold}_{3,3}^{\mathrm{reg}} = \mathrm{Mold}_{3,3}^{\mathrm{D}_3} \coprod \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1} \coprod \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2} \coprod \mathrm{Mold}_{3,3}^{\mathrm{J}_3} \coprod \mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$$

such that

(1) $\mathrm{Mold}_{3,3}^{\mathrm{D}_3}$ *is a smooth integral scheme of relative dimension* 6 *over* $\mathbb{Z}$.
(2) $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1}$ *is a smooth integral scheme of relative dimension* 5 *over* $\mathbb{Z}[1/2]$.
(3) $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2}$ *is a smooth variety of dimension* 5 *over* $\mathbb{F}_2$.

(4) $\mathrm{Mold}_{3,3}^{\mathrm{J}_3}$ *is a smooth integral scheme of relative dimension* 4 *over* $\mathbb{Z}[1/3]$.

(5) $\mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$ *is a smooth variety of dimension* 4 *over* $\mathbb{F}_3$.

Remark 30. For the definitions of M_3^* and $\mathrm{Mold}_{3,3}^*$ for $* = \mathrm{D}_3, \mathrm{C}_2 \times \mathrm{D}_1, \mathrm{J}_3$, and so on in Theorems 28 and 29, see [6]. Geometric points of $\mathrm{Mold}_{3,3}^*$ correspond to subalgebras of type $*$. The smooth morphism $\psi : \mathrm{M}_3^{\mathrm{reg}} \to \mathrm{Mold}_{3,3}^{\mathrm{reg}}$ induces a smooth morphism $\psi_* : \mathrm{M}_3^* \to \mathrm{Mold}_{3,3}^*$ for each $*$.

Let $V = \mathcal{O}_{\mathbb{Z}}^{\oplus 3}$ be a free sheaf of rank 3 on $\operatorname{Spec} \mathbb{Z}$. Let us denote by $\mathbb{P}_*(V)$ and $\mathbb{P}^*(V)$ the projective spaces consisting of rank 1 and rank 2 subbundles of V, respectively.

Let us define $\varphi_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \to \mathrm{Mold}_{3,3}$ by $(W_1, W_2) \mapsto \langle \mathrm{Hom}(V/W_1, W_2)\rangle$, where $\langle \mathrm{Hom}(V/W_1, W_2)\rangle$ is the subalgebra of $\mathrm{Hom}(V,V)$ generated by $\mathrm{Hom}(V/W_1, W_2)$.

Let us define $\varphi_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \to \mathrm{Mold}_{3,3}$ by $(L_1, L_2) \mapsto \langle \mathrm{Hom}(V/L_1, L_2)\rangle$, where $\langle \mathrm{Hom}(V/L_1, L_2)\rangle$ is the subalgebra of $\mathrm{Hom}(V,V)$ generated by $\mathrm{Hom}(V/L_1, L_2)$.

Here we regard $f \in \mathrm{Hom}(V/W_1, W_2)$ as an element of $\mathrm{Hom}(V,V)$ by $V \overset{\text{proj.}}{\to} V/W_1 \overset{f}{\to} W_2 \hookrightarrow V$. We also regard $f \in \mathrm{Hom}(V/L_1, L_2)$ as an element of $\mathrm{Hom}(V,V)$ in the same way.

Theorem 31 ([6])**.** *Let* $\varphi'_{\mathrm{S}_2} : \mathbb{P}^*(V) \times \mathbb{P}^*(V) \setminus \Delta \to \mathrm{Mold}_{3,3}$ *and* $\varphi'_{\mathrm{S}_3} : \mathbb{P}_*(V) \times \mathbb{P}_*(V) \setminus \Delta \to \mathrm{Mold}_{3,3}$ *be the induced morphisms by* φ_{S_2} *and* φ_{S_3}*, respectively. Here we denote by* Δ *the diagonal of* $\mathbb{P}^*(V) \times \mathbb{P}^*(V)$ *or* $\mathbb{P}_*(V) \times \mathbb{P}_*(V)$*. Then* φ'_{S_2} *and* φ'_{S_3} *are smooth.*

From the theorem above, we can define $\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$.

Definition 32 ([6])**.** We define open subschemes $\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$ of $\mathrm{Mold}_{3,3}$ as $\mathrm{Mold}_{3,3}^{\mathrm{S}_2} := \varphi_{\mathrm{S}_2}(\mathbb{P}^*(V) \times \mathbb{P}^*(V) \setminus \Delta)$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3} := \varphi_{\mathrm{S}_3}(\mathbb{P}_*(V) \times \mathbb{P}_*(V) \setminus \Delta)$, respectively.

Remark 33. Geometric points of $\mathrm{Mold}_{3,3}^{\mathrm{S}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_3}$ correspond to subalgebras of type S_2 and S_3, respectively. Let $\mathrm{Mold}_{3,3}^{\text{non-comm}}$ be the open subscheme of $\mathrm{Mold}_{3,3}$ consisting of non-commutative subalgebras. Then $\mathrm{Mold}_{3,3}^{\text{non-comm}} = \mathrm{Mold}_{3,3}^{\mathrm{S}_2} \coprod \mathrm{Mold}_{3,3}^{\mathrm{S}_3}$.

Note that
$$\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} = \varphi_{\mathrm{S}_2}(\mathbb{P}^*(V) \times \mathbb{P}^*(V))$$
and
$$\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}} = \varphi_{\mathrm{S}_3}(\mathbb{P}_*(V) \times \mathbb{P}_*(V)).$$

Now we can state the following theorem.

Theorem 34 ([6])**.** *There is an irreducible decomposition*
$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cup \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}},$$
where the relative dimensions of $\overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}}$, $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *over* $\mathbb{Z}$ *are* 6, 4, *and* 4, *respectively. Moreover, both* $\mathrm{Mold}_{3,3}^{\mathrm{S}_5} := \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}}$ *and* $\mathrm{Mold}_{3,3}^{\mathrm{S}_4} := \overline{\mathrm{Mold}_{3,3}^{\mathrm{reg}}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}}$ *have relative dimension* 2 *over* $\mathbb{Z}$, *and* $\overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_2}} \cap \overline{\mathrm{Mold}_{3,3}^{\mathrm{S}_3}} = \emptyset$.

The figure below shows the relation among $\mathrm{Mold}_{3,3}^{*}$ ($* = \mathrm{D}_3, \mathrm{N}_2 \times \mathrm{D}_1, \mathrm{J}_3$, and so on) in $\mathrm{Mold}_{3,3}$. To be exact, $\mathrm{N}_2 \times \mathrm{D}_1$ and J_3 denote $\mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1} \cup \mathrm{Mold}_{3,3}^{\mathrm{N}_2\times\mathrm{D}_1/\mathbb{F}_2}$ and $\mathrm{Mold}_{3,3}^{\mathrm{J}_3} \cup \mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$, respectively. Two moduli schemes are connected with an edge if the closure of the upper moduli contains the lower. For example, the closure of $\mathrm{Mold}_{3,3}^{\mathrm{J}_3} \cup \mathrm{Mold}_{3,3}^{\mathrm{J}_3/\mathbb{F}_3}$ contains $\mathrm{Mold}_{3,3}^{\mathrm{S}_5}$ and $\mathrm{Mold}_{3,3}^{\mathrm{S}_4}$.

$$\mathrm{D}_3 = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\}$$

$$|$$

$$\mathrm{N}_2 \times \mathrm{D}_1 = \left\{ \begin{pmatrix} a & c & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$$

$$|$$

$$\mathrm{S}_2 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \right\} \qquad \mathrm{J}_3 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \right\} \qquad \mathrm{S}_3 = \left\{ \begin{pmatrix} a & 0 & c \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \right\}$$

$$\backslash \qquad\qquad / \qquad\qquad \backslash \qquad\qquad /$$

$$\mathrm{S}_5 = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} \right\} \qquad \mathrm{S}_4 = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \right\}$$

The moduli schemes $\mathrm{Mold}_{3,4}$ and $\mathrm{Mold}_{3,5}$ will be discussed in [6].

5. Appendix

In this appendix, we show results in the degree 2 case.

Proposition 35 ([6]). *Any subalgebras of* $\mathrm{M}_2(k)$ *can be classified into one of the following cases:*

(1) $\mathrm{M}_2(k)$

(2) $\mathrm{B}_2(k) := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$

(3) $\mathrm{D}_2(k) := \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$

(4) $\mathrm{N}_2(k) := \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \middle| \, a, b \in k \right\}$

(5) $\mathrm{C}_2(k) := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| \, a \in k \right\}$

Example 36 ([4, Example 1.1])**.** In the case $n = 2$, we have

$$\begin{aligned}
\mathrm{Mold}_{2,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{2,2} &= \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{2,3} &= \mathbb{P}^1_{\mathbb{Z}}, \\
\mathrm{Mold}_{2,4} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

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