LOCALIZATION FUNCTORS IN DERIVED CATEGORIES OF COMMUTATIVE NOETHERIAN RINGS

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Abstract. We report some results about localization functors on the unbounded derived category of a commutative Noetherian ring. In particular, we give a new way to calculate localization functors by the notion of Čech complexes.

1. Introduction

This article is based on joint work with Yuji Yoshino [9].

Let R be a commutative Noetherian ring, and $\operatorname{Mod} R$ be the category of all R-modules. We denote by $\mathcal{D} = D(\operatorname{Mod} R)$ the unbounded derived category of $\operatorname{Mod} R$. For a triangulated subcategory $\mathcal{T}$ of $\mathcal{D}$, its left (resp. right) orthogonal subcategory is defined as ${}^{\perp}\mathcal{T} = \{\, X \in \mathcal{D} \,|\, \operatorname{Hom}_{\mathcal{D}}(X, \mathcal{T}) = 0 \,\}$ (resp. $\mathcal{T}^{\perp} = \{\, X \in \mathcal{D} \,|\, \operatorname{Hom}_{\mathcal{D}}(\mathcal{T}, X) = 0 \,\}$). Furthermore, $\mathcal{T}$ is called localizing (resp. colocalizing) if $\mathcal{T}$ is closed under arbitrary direct sums (resp. products).

The support of a complex $X \in \mathcal{D}$ is defined as

$$\operatorname{supp} X = \{\, \mathfrak{p} \in \operatorname{Spec} R \,|\, X \otimes^{\mathrm{L}}_R \kappa(\mathfrak{p}) \neq 0 \,\},$$

where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. For a subset $W \subseteq \operatorname{Spec} R$, it is seen that the full subcategory $\mathcal{L}_W = \{\, X \in \mathcal{D} \,|\, \operatorname{supp} X \subseteq W \,\}$ is localizing. In [10], Neeman proved the equality

$$\mathcal{L}_W = \operatorname{Loc}\{\, \kappa(\mathfrak{p}) \,|\, \mathfrak{p} \in W \},$$

where $\operatorname{Loc}\{\, \kappa(\mathfrak{p}) \,|\, \mathfrak{p} \in W \}$ denotes the smallest localizing subcategory containing the set $\{\, \kappa(\mathfrak{p}) \,|\, \mathfrak{p} \in W \}$. Since $\mathcal{L}_W$ is generated by a small set, there is a right adjoint $\gamma_W : \mathcal{D} \to \mathcal{L}_W$ to the inclusion functor $i_W : \mathcal{L}_W \hookrightarrow \mathcal{D}$, see [7]. At the same time, we obtain a left adjoint $\lambda_W : \mathcal{D} \to \mathcal{L}_W^{\perp}$ to the inclusion functor $j_W : \mathcal{L}_W^{\perp} \hookrightarrow \mathcal{D}$.

The functor $\gamma_W = (i_W \gamma_W)$ is a colocalization on $\mathcal{D}$, that is, there is a morphism $\varepsilon : \gamma_W \to \mathrm{id}_{\mathcal{D}}$ such that $\gamma_W \varepsilon$ is invertible, and the equality $\gamma_W \varepsilon = \varepsilon \gamma_W$ holds. Furthermore, $\lambda_W (= j_W \lambda_W)$ is a localization on $\mathcal{D}$, that is, there is a morphism $\eta : \mathrm{id}_{\mathcal{D}} \to \lambda_W$ such that $\lambda_W \eta$ is invertible, and the equality $\lambda_W \eta = \eta \lambda_W$ holds. This notion originally appeared in a topological work by Bousfield [2], and they play a significant role in such a field.

We remark that, in general, γ_W and λ_W are constructed by using Brown representation theorem. For this reason, it is not easy to know the form of γ_W and λ_W. However, there are some cases in which we can describe (co)localization functors as derived functors of R-linear functors on $\operatorname{Mod} R$. We shall give such examples in the next section.

The detailed version of this paper will be submitted for publication elsewhere.

2. Examples of (co)localization functors

Let W be a specialization-closed (resp. generalization-closed) subset of $\operatorname{Spec} R$, that is, if $\mathfrak{p}$ is a prime ideal with $\mathfrak{q} \subseteq \mathfrak{p}$ (resp. $\mathfrak{p} \subseteq \mathfrak{q}$) for some $\mathfrak{q} \in W$, then $\mathfrak{p}$ belongs to W. For an ideal $\mathfrak{a} \subseteq R$ and a multiplicatively closed subset $S \subseteq R$, we write $V(\mathfrak{a}) = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{a} \subseteq \mathfrak{p} \}$ and $U_S = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \emptyset \}$. Clearly, $V(\mathfrak{a})$ is specialization-closed, and U_S is generalization-closed.

Let V be a specialization-closed subset of $\operatorname{Spec} R$. It is well-known that there is an isomorphism

$$\gamma_V \cong \mathrm{R}\Gamma_V,$$

where $\mathrm{R}\Gamma_V$ is the right derived functor of the section functor $\Gamma_V : \operatorname{Mod} R \to \operatorname{Mod} R$ with support in V; it induces the local cohomology functors $H^i_V(-) = H^i(\mathrm{R}\Gamma_V(-))$.

As a natural generalization of this fact, the author and Yoshino proved in [8, Proposition 3.1] that if $W = V \cap U_S$ for a multiplicatively closed subset S, then

$$\gamma_W \cong \mathrm{R}\Gamma_V \mathrm{RHom}_R(S^{-1}R, -),$$

which is the right derived functor of the left exact functor $\Gamma_V \mathrm{Hom}_R(S^{-1}R, -)$ on $\operatorname{Mod} R$. In particular, when $V = \operatorname{Spec} R$, we obtain an isomorphism

$$\gamma_{U_S} \cong \mathrm{RHom}_R(S^{-1}R, -),$$

see also [4, p. 175].

For a prime ideal $\mathfrak{p}$ of R, we write $U(\mathfrak{p}) = \{ \mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p} \}$. It then follows that $U(\mathfrak{p}) = U_S$ for $S = R \backslash \mathfrak{p}$. Since $V(\mathfrak{p}) \cap U(\mathfrak{p}) = \{\mathfrak{p}\}$, as a special case of the above fact, we get an isomorphism

$$\gamma_{\{\mathfrak{p}\}} \cong \mathrm{R}\Gamma_{V(\mathfrak{p})} \mathrm{RHom}_R(R_{\mathfrak{p}}, -).$$

For a subset W of $\operatorname{Spec} R$, $\dim W$ denotes the supremum of lengths of chains of distinct prime ideals in W. It is possible to extend the above isomorphism about $\gamma_{\{\mathfrak{p}\}}$ to the case of arbitrary subsets W with $\dim W = 0$. In such a case, the following isomorphism holds;

$$\gamma_W \cong \bigoplus_{\mathfrak{p} \in W} \gamma_{\{\mathfrak{p}\}} \cong \bigoplus_{\mathfrak{p} \in W} \mathrm{R}\Gamma_{V(\mathfrak{p})} \mathrm{RHom}_R(R_{\mathfrak{p}}, -),$$

see [8, Theorem 3.12].

Let S be a multiplicatively closed subset S of R. The classical localization with respect to S is a typical example of localization functors on $\mathcal{D}$. In fact, it is easy to see that $(-) \otimes_R S^{-1}R : \mathcal{D} \to \mathcal{D}$ satisfies the definition of localization functors. We can also describe the functor $(-) \otimes_R S^{-1}R$ by our notation λ_W. However, before that, it might be better to introduce another notation from the viewpoint of cosupport.

The cosupport of $X \in \mathcal{D}$ is defined as

$$\operatorname{cosupp} X = \{ \mathfrak{p} \in \operatorname{Spec} R \mid \mathrm{RHom}_R(\kappa(\mathfrak{p}), X) \neq 0 \}.$$

For a subset W of $\operatorname{Spec} R$, we write $\mathcal{C}^W = \{ X \in \mathcal{D} \mid \operatorname{cosupp} X \subseteq W \}$, which is a colocalizing subcategory of $\mathcal{D}$. Then we can show the following equality;

$$\mathcal{C}^W = \mathcal{L}_{W^c}^{\perp},$$

where $W^c = \operatorname{Spec} R \backslash W$. Recall that $\lambda_{W^c} : \mathcal{D} \to \mathcal{L}_{W^c}^{\perp}$ is a left adjoint to the inclusin functor $\mathcal{L}_{W^c}^{\perp} \hookrightarrow \mathcal{D}$. In terms of the equality $\mathcal{C}^W = \mathcal{L}_{W^c}^{\perp}$, we write

$$\lambda^W = \lambda_{W^c}.$$

In other words, λ^W is a left adjoint to the inclusion functor $\mathcal{C}^W \hookrightarrow \mathcal{D}$. Under this notation, it holds that

$$\lambda^{U_S} \cong (-) \otimes_R S^{-1}R.$$

There is another important example of localization functors. Let $\mathfrak{a}$ be an ideal of R. Greenlees and May [5] proved that the left derived functor $\mathrm{L}\Lambda^{V(\mathfrak{a})}$ of the $\mathfrak{a}$-adic completion functor $\Lambda^{V(\mathfrak{a})} : \operatorname{Mod} R \to \operatorname{Mod} R$ is a right adjoint to $\mathrm{R}\Gamma_{V(\mathfrak{a})}$, see also [1]. The functor $H_i^{\mathfrak{a}}(-) = H^{-i}(\mathrm{L}(\Lambda^{V(\mathfrak{a})}(-))$ is called the ith local homology functor with respect to $\mathfrak{a}$. Using the adjointness property of $\mathrm{R}\Gamma_{V(\mathfrak{a})}$ and $\mathrm{L}\Lambda^{V(\mathfrak{a})}$, we can prove that

$$\lambda^{V(\mathfrak{a})} \cong \mathrm{L}\Lambda^{V(\mathfrak{a})},$$

see [8, Proposition 5.1]. Furthermore, if $W = V(\mathfrak{a}) \cap U_S$, then

$$\lambda^W \cong \mathrm{L}\Lambda^{V(\mathfrak{a})}(- \otimes_R S^{-1}R),$$

see [9, Corollary 3.6]. Hence, for a prime ideal $\mathfrak{p}$, we have

$$\lambda^{\{\mathfrak{p}\}} \cong \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}).$$

In addition, if W is an arbitrary subset W with $\dim W = 0$, then it holds that

$$\lambda^W \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} \cong \prod_{\mathfrak{p} \in W} \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}),$$

see [9, Theorem 3.10].

In the next section, we give a method to compute γ_W and λ^W for subsets $W \subseteq \operatorname{Spec} R$ with $\dim W > 0$.

3. Mayer-Vietoris triangles

Let W be a subset of $\operatorname{Spec} R$. We denote by $\varepsilon_W : \gamma_W \to \mathrm{id}_{\mathcal{D}}$ and $\eta^W : \mathrm{id}_{\mathcal{D}} \to \lambda^W$ the natural morphisms. Note that when $W' \subseteq W$, there are isomorphisms $\gamma_{W'}\gamma_W \cong \gamma_{W'}$ and $\lambda^{W'}\lambda^W \cong \lambda^{W'}$. This fact implicitly used in the theorem below.

We say that a subset W' of W is specialization-closed (resp. generalization-closed) in W if the inclusion relation $V(\mathfrak{p}) \cap W \subseteq W'$ (resp. $U(\mathfrak{p}) \cap W \subseteq W'$) holds for any $\mathfrak{p} \in W'$.

Theorem 1 ([9, Theorem 3.15, Theorem 3.21])**.** *Let W, W_0 and W_1 be subsets of* $\operatorname{Spec} R$ *with $W = W_0 \cup W_1$. Suppose that one of the following conditions holds:*

(1) *W_0 is specialization-closed in W;*
(2) *W_1 is generalization-closed in W.*

Then, for any $X \in \mathcal{D}$, there are triangles of the following form;

$$\begin{array}{ccccccc} \gamma_{W_1}\gamma_{W_0}X & \xrightarrow{a} & \gamma_{W_1}X \oplus \gamma_{W_0}X & \xrightarrow{b} & \gamma_W X & \longrightarrow & \gamma_{W_1}\gamma_{W_0}X[1], \\ \lambda^W X & \xrightarrow{c} & \lambda^{W_1}X \oplus \lambda^{W_0}X & \xrightarrow{d} & \lambda^{W_1}\lambda^{W_0}X & \longrightarrow & \lambda^W X[1], \end{array}$$

where a, b, c and d are the morphisms represented by the following matrices:

$$a = \begin{pmatrix} (-1)\cdot \gamma_{W_1}\varepsilon_{W_0}X \\ \varepsilon_{W_1}\gamma_{W_0}X \end{pmatrix} \quad , \quad b = \begin{pmatrix} \varepsilon_{W_1}\gamma_W X & \varepsilon_{W_0}\gamma_W X \end{pmatrix}$$

$$c = \begin{pmatrix} \eta^{W_1}\lambda^W X \\ \eta^{W_0}\lambda^W X \end{pmatrix} \quad , \quad d = \begin{pmatrix} \lambda^{W_1}\eta^{W_0}X & (-1)\cdot \eta^{W_1}\lambda^{W_0}X \end{pmatrix}$$

By this theorem, we can compute γ_W (resp. λ^W) by using γ_{W_0} and γ_{W_1} (resp. λ^{W_0} and λ^{W_1}) for smaller subsets W_0 and W_1. Furthermore, as long as we work on $\mathcal{D}$, the theorem generalizes Mayer-Vietoris triangles in the sense of Benson, Iyengar and Krause [3, Theorem 7.5], see also [9, Remark 3.23].

By Theorem 1, it is possible to give a simpler proof of a classical theorem due to Gruson and Raynaud [6, II; Corollary 3.2.7], which states that the projective dimension of any flat R-module is at most the the Krull dimension of R, see [9, §4].

We give an example of Theorem 1.

Example 2. Let $(R, \mathfrak{m})$ be a 1-dimensional local domain with quotient field Q. Put $W = \operatorname{Spec} R$, $W_0 = \{\mathfrak{m}\}$ and $W_1 = \{(0)\}$. Set $X = R$. We denotes by $\widehat{R}$ the $\mathfrak{m}$-adic completion of R. Then the second triangle of Theorem 1 yields a short exact sequence of R-modules;

$$0 \longrightarrow R \longrightarrow Q \oplus \widehat{R} \longrightarrow \widehat{R} \otimes_R Q \longrightarrow 0.$$

One may notice from Theorem 1 and this example that γ_W and λ^W can be computed by the notion of Čech complexes. In the final section, we give a sketch of this fact for λ^W.

4. Čech complexes

In this section, we suppose that $n = \dim R$ is finite. For $W \subseteq \operatorname{Spec} R$ with $\dim W = 0$, we define a functor $\bar{\lambda}^W : \operatorname{Mod} R \to \operatorname{Mod} R$ by

$$\bar{\lambda}^W = \prod_{\mathfrak{p}\in W} \Lambda^{V(\mathfrak{p})}(- \otimes R_{\mathfrak{p}}).$$

For $\mathfrak{p} \in \operatorname{Spec} R$, we denote by $\bar{\eta}^{\{\mathfrak{p}\}} : \mathrm{id}_{\operatorname{Mod} R} \to \bar{\lambda}^{\{\mathfrak{p}\}} = \Lambda^{V(\mathfrak{p})}(- \otimes R_{\mathfrak{p}})$ the composition of the canonical morphisms $\mathrm{id}_{\operatorname{Mod} R} \to (-) \otimes R_{\mathfrak{p}}$ and $(-) \otimes R_{\mathfrak{p}} \to \Lambda^{V(\mathfrak{p})}(- \otimes R_{\mathfrak{p}})$. Moreover, $\bar{\eta}^W : \mathrm{id}_{\operatorname{Mod} R} \to \bar{\lambda}^W = \prod_{\mathfrak{p}\in W} \bar{\lambda}^{\{\mathfrak{p}\}}$ denotes the product of the morphisms $\bar{\eta}^{\{\mathfrak{p}\}}$ for $\mathfrak{p} \in W$.

Let W be an arbitrary subset of $\operatorname{Spec} R$. Put $W_i = \{\, \mathfrak{p} \in W \mid \dim R/\mathfrak{p} = i \,\}$ for $0 \le i \le n$, and write $\mathbb{W} = \{W_i\}_{0\le i\le n}$. It is possible to construct a Čech complex of the following form;

$$\prod_{0\le i\le n} \bar{\lambda}^{W_i} \longrightarrow \prod_{0\le i<j\le n} \bar{\lambda}^{W_j}\bar{\lambda}^{W_i} \longrightarrow \cdots \longrightarrow \bar{\lambda}^{W_n}\cdots\bar{\lambda}^{W_0},$$

see [9, §7]. We denote by $L^{\mathbb{W}}$ this complex of functors. It sends a complex X of R-module to a double complex in a natural way. We write $\mathrm{tot}L^{\mathbb{W}}X$ for the total complex of the double complex. Under this setting, we can prove the following theorem.

Theorem 3 ([9, Corollary 7.9, Proposition 8.5])**.** *Let W and $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ be as above. Let $X \in \mathcal{D}$, and suppose that one of the following conditions holds:*

(1) *X consists of flat R-modules;*
(2) *X consists of finitely generated R-modules.*

Then there is an isomorphism in $\mathcal{D}$;

$$\lambda^W X \cong \mathrm{tot} L^{\mathbb{W}} X.$$

The case (1) of this theorem is proved by using Theorem 1. The case (2) is deduced from the case (1) and the following isomorphisms for a complex X of finitely generated R-modules;

$$\lambda^W X \cong (\lambda^W R) \otimes_R^{\mathrm{L}} X \cong (L^{\mathbb{W}} R) \otimes_R X \cong \mathrm{tot}\, L^{\mathbb{W}} X.$$

See [9, §7, §8] for more details.

In the both case of (1) and (2), $\mathrm{tot} L^{\mathbb{W}} X$ consists of pure-injective R-modules. Therefore, noting that $X \cong \lambda^W X$ for $W = \operatorname{Spec} R$, one can get a functorial way to construct pure-injective resolutions by Theorem 3, see [9, §9].

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