ON QUIVER GRASSMANNIANS AND ORBIT CLOSURES FOR GEN-FINITE MODULES

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Abstract. We call an A-module M gen-finite if there are only finitely many indecomposable modules generated by M. We construct desingularisations for orbit closures and quiver Grassmannians of gen-finite modules. This generalises previous work of Crawley-Boevey and the second author. The construction uses a tilt of the endomorphism ring of a cogenerator containing as summand all indecomposable modules generated by M.

1. Introduction

In this article, we will only consider finite-dimensional left modules over finite-dimensional algebras over a field $\mathbb{K}$. We give a construction of two desingularisations, one of orbit closures for gen-finite modules (i.e. modules generating only finitely many isomorphism classes of indecomposables) and one of quiver Grassmannians of gen-finite modules. The main idea is to use a *canonical* tilt of a cogenerator having as a summand all indecomposable modules generated by the gen-finite module. For orbit closures, the first appearance of this construction is in [4] for modules over the algebra $\mathbb{K}[T]/T^n$, then at least partly in [2] for path algebras of a Dynkin quiver, and in [3] for representation-finite algebras. A different construction for gen-finite modules appears in Zwara.

For quiver Grassmannians, the first appearance of a special case of the desingularisation is in [1] for modules over path algebras of a Dynkin quiver and then in [3] for representation-finite algebras. Our motivation for this generalisation of [3] was that all finiteness assumptions had been made for the algebra but the varieties (orbit closures and quiver Grassmannians) are defined for a specific module—therefore we looked for a finiteness criterion on the module for which the same results can be proven. This finiteness criterion is gen-finiteness.

2. Algebraic maps from an idempotent element

Let B be a finite-dimensional algebra, $e \in B$ an idempotent element and $A = eBe$. We obtain from e a diagram

$$\begin{tikzcd}[column sep=huge] B/BeB\text{-mod} \arrow[r, "i" description] & B\text{-mod} \arrow[l, shift right=3, "q=B/BeB\otimes_B(-)"'] \arrow[l, shift left=3, "p=\mathrm{Hom}_B(B/BeB,-)"] \arrow[r, "e" description] & A\text{-mod} \arrow[l, shift right=3, "\ell=Be\otimes_A -"'] \arrow[l, shift left=3, "r=\mathrm{Hom}_A(eB,-)"] \end{tikzcd} \tag{2.1}$$

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of six functors. Such data is known as a recollement of abelian categories, and can be defined in abstract, but we will only consider recollements of module categories determined by idempotents as above. Since ℓ and r are fully faithful, there is a natural transformation $\ell \to r$. We define the *intermediate extension* functor to be $c = \operatorname{im}(\ell \to r)$ (cf. [5]).

Definition 2.1. For $X \in B$-mod we define full subcategories of B-mod

$$\operatorname{gen}(X) = \{Z \mid \exists \text{ exact } X^n \to Z \to 0\},$$
$$\operatorname{gen}^1(X) = \{Z \mid \exists \text{ exact } X_1 \to X_0 \to Z \to 0,\ X_i \in \operatorname{add} X : \operatorname{Hom}_B(X, X_1) \to \operatorname{Hom}_B(X, X_0) \to \operatorname{Hom}_B(X, Z) \to 0 \text{ is exact}\}.$$

We define $\operatorname{cogen}(X)$ and $\operatorname{cogen}^1(X)$ dually.

Lemma 2.2. *In the context of the idempotent recollement* (2.1)*, write $P = Be$ and $I = \mathrm{D}(eB)$. Then*

$$\ker q = \operatorname{gen}(P) \supseteq \operatorname{gen}_1(P) = \operatorname{im} \ell,$$
$$\ker p = \operatorname{cogen}(I) \supseteq \operatorname{cogen}^1(I) = \operatorname{im} r.$$

Moreover, the image of the intermediate extension $c = \operatorname{im}(\ell \to r)$ is given by

$$\operatorname{im} c = \ker p \cap \ker q = \operatorname{gen}(P) \cap \operatorname{cogen}(I).$$

In the context of (2.1), we call the elements in $\ker p = \operatorname{cogen}(I)$ *stable* modules.

Now let $A = \mathbb{K}Q/I$, where $\mathbb{K}$ is an algebraically closed field, Q is a finite quiver with n vertices, and I is an admissible ideal of $\mathbb{K}Q$. For $d \in \mathbb{Z}^n_{\geq 0}$, we denote by

$$R_A(d) = \{M \in \prod_{(i \to j) \in Q_1} \operatorname{Hom}_{\mathbb{K}}(\mathbb{K}^{d_i}, \mathbb{K}^{d_j}) \mid IM = 0\}$$

the representation space of d-dimensional A-modules. It is an affine variety and carries a natural action of the algebraic group $\mathbf{Gl}_d := \prod_{i=1}^n \mathbf{Gl}_{d_i}$. The orbits correspond to isomorphism classes of d-dimensional A-modules.

Now assume A fits into a recollement as in (2.1). Then, choosing a complete set of primitive orthogonal idempotents of B extending that of A, we can write the dimension vector of a B-module X as $(d, s) \in \mathbb{Z}^n \times \mathbb{Z}^{t-n}$, where $d = \underline{\dim}\, eX$ and $s = \underline{\dim}(1-e)X$. We number the components of d from 1 to n, and those of s from $n+1$ to t. We have $\mathbf{Gl}_{(d,s)} = \mathbf{Gl}_d \times \mathbf{Gl}_s$, so it makes sense to consider only the $\mathbf{Gl}_s$ action on $R_B(d, s)$. Now the restriction functor e provides a regular map

$$e\colon R_B(d, s) \to R_A(d).$$

Using B, e, and the associated intermediate extension functor c, we give constructions of candidate desingularisations as follows.

Orbit closures: Pick a d-dimensional A-module M and write $\overline{\mathcal{O}}_M \subset R_A(d)$ for its $\mathbf{Gl}_d$-orbit closure. Set $(d, s) = \underline{\dim}\, c(M)$ and let $\overline{\mathcal{O}}^{\mathrm{st}}_{c(M)} \subset R_B(d, s)$ be the set of stable points in the orbit closure $\overline{\mathcal{O}}_{c(M)}$ (this is a Zariski-open subset). On this open subset the $\mathbf{Gl}_s$-operation is free, therefore the geometric quotient exists. Our candidate for the desingularisation of an orbit closure is then the map

$$\pi\colon \overline{\mathcal{O}}^{\mathrm{st}}_{c(M)}/\mathbf{Gl}_s \to \overline{\mathcal{O}}_M$$

induced by e. Since we will use GIT methods to show that this is a projective map, we will add the extra assumption that $\mathrm{char}(\mathbb{K}) = 0$ in this case.

Quiver Grassmannians: Now, let M be an A-module and $d \in \mathbb{Z}^n_{\geq 0}$ be any dimension vector. The quiver Grassmannian of d-dimensional A-submodules is defined as

$$\mathrm{Gr}_A\binom{M}{d} = \{U \subset M \mid U \text{ an } A\text{-submodule of } M,\ \underline{\dim}\, U = d\}.$$

This is a projective variety (we will only consider the reduced scheme structure on it). Instead of orbits, for $N \in A$-mod we define the following locally closed subsets

$$\mathcal{E}^{[N]} := \{U \in \mathrm{Gr}_A\tbinom{M}{d} \mid M/U \cong N\}.$$

This is a locally closed irreducible subset of $\mathrm{Gr}_A\binom{M}{d}$, it is non-empty if and only if there is an epimorphism $M \to N$, and in this case it is smooth of dimension $\dim_{\mathbb{K}} \mathrm{Hom}_A(M,N) - \dim_{\mathbb{K}} \mathrm{End}_A(N)$. Assuming that there are only finitely many of these strata, then there are modules $N_1, \ldots, N_t$ such that $\mathrm{Gr}_A\binom{M}{d} = \bigcup_{i=1}^t \overline{\mathcal{E}}^{[N_i]}$ is a decomposition into irreducible components.

Now write $(d, s_i) = \underline{\dim}\, c(M) - \underline{\dim}\, c(N_i)$, and consider the algebraic map

$$\mathrm{Gr}_B\binom{c(M)}{d,s_i} \to \mathrm{Gr}_A\binom{M}{d}$$

induced by e. This is a projective map since it is an algebraic map between projective varieties, and it restricts to a projective map

$$p_i \colon \overline{\mathcal{E}}^{[c(N_i)]} \to \overline{\mathcal{E}}^{[N_i]}.$$

Combining the various maps p_i, we obtain a dominant projective map

$$p = \bigsqcup_{i=1}^{t} p_i \colon \bigsqcup_{i=1}^{t} \overline{\mathcal{E}}^{[c(N_i)]} \to \mathrm{Gr}_A\binom{M}{d}.$$

This is our candidate for the desingularisation of the quiver Grassmannian. The rest of this article will explain the following: given a gen-finite module M, then we can find B and e such that π and p are desingularisations.

3. A tilting module on endomorphism rings of cogenerators

Let $E \in A$-mod be a basic module with $E = \mathrm{D}A \oplus X$ for some module X (i.e. E is a basic cogenerator). Let $\Gamma = \mathrm{End}_A(E)^{\mathrm{op}}$. We remark that for every faithful projective module P there exists a classical (i.e. of projective dimension at most 1) tilting module T_P such that $\mathrm{gen}(P) = \mathrm{gen}(T_P)$. This follows since $\mathrm{gen}(P)$ is a faithful torsion class.

Lemma 3.1. *The module $P = \mathrm{Hom}_A(E, \mathrm{D}A)$ is a faithful projective (left) Γ-module.*

We write T_P for the basic classical tilting Γ-module with $\mathrm{gen}(T_P) = \mathrm{gen}(P)$ and observe that $T_P = P \oplus Y$ for some module Y. We write $B = \mathrm{End}_\Gamma(T_P)^{\mathrm{op}}$ for the tilted algebra and $e \in B$ denote the projection onto the summand P of T_P. Then $C := \mathrm{D}T_P$ is a classical cotilting (left) B-module. We observe $A = \mathrm{End}_A(A)^{\mathrm{op}} \cong \mathrm{End}_\Gamma(P)^{\mathrm{op}} = eBe$.

Theorem 3.2. *With the notation of the preceding paragraph, the intermediate extension functor c associated to e satisfies*

$$c(\mathrm{D}A) = \mathrm{D}(eB), \quad c(E) = C.$$

The key step in the proof is that B is the endomorphism ring of

$$(E \to I(E)) \oplus (0 \to \mathrm{D}A),$$

viewed as a 2-term complex in the homotopy category of A, where $E \to I(E)$ denotes a minimal injective envelope of E.

We will refer to the algebra B as the *cogenerator-tilted algebra* (for the cogenerator E) and call e the *special idempotent*.

4. Desingularisation of orbit closures

We first realise arbitrary rank varieties of A as affine quotient varieties, using B and e as constructed above. The description of B in terms of the homotopy category of A provides us with a connection between the representation space of B and rank varieties in the representation space $R_A(d)$ of A-modules with dimension vector d. For finite-dimensional A-modules X and Y, write $[X,Y] = \dim \mathrm{Hom}_A(X,Y)$. Write $Q_0 = \{1,\dots,n\}$ and e_i for the primitive idempotent corresponding to $i \in Q_0$. Let $\underline{\dim}\, M = (\dim e_i M)_{1\le i\le n}$ be the dimension vector of an A-module M. Given two A-modules M and N, we write $\underline{\dim}\, M \le \underline{\dim}\, N$ if this inequality holds componentwise.

Let $E = \bigoplus_{i=1}^t E_i$ for $E_i \in A$-mod indecomposable, and let $m = (m_1,\dots,m_t) \in \mathbb{Z}^t_{\ge 0}$. We define the *rank variety* as

$$\mathcal{C}^E_m := \{N \in R_A(d) \mid [N, E_i] \ge m_i,\ 1 \le i \le t\}.$$

Since the map $R_A(d) \to \mathbb{Z}_{\ge 0}$ defined by $N \mapsto [N,X]$ is upper-semicontinuous for every module X, the subset $\mathcal{C}^E_m$ is Zariski-closed in $R_A(d)$. For any fixed module $M \in R_A(d)$, we write $\mathcal{C}^E_M := \mathcal{C}^E_m$ where $m_i = [M, E_i]$.

Let B be the cogenerator-tilted algebra of E, and let e be its special idempotent. Since $A \cong eBe$, we may choose a complete set of primitive orthogonal idempotents of B extending that of A, and thus write the dimension vector of a B-module X as $(d,s) \in \mathbb{Z}^n \times \mathbb{Z}^{t-n}$ as before. The map $e\colon R_B(d,s) \to R_A(d)$ induces by [3, Lem. 6.3] an isomorphism of varieties

$$R_B(d,s) /\!\!/ \mathbf{Gl}_s \xrightarrow{\sim} \mathrm{im}\, e.$$

Furthermore, $\mathrm{im}\, e = \{N \in R_A(d) \mid \underline{\dim}\, c(N) \le (d,s)\}$ is a closed subset of $R_A(d)$ [3, Lem. 7.2]. For any injective A-module I and any dimension vector d, let

$$[d, I] := [N, I]$$

where $N \in R_A(d)$ is arbitrary, noting that $[N,I]$ depends only on $\underline{\dim}\, N = d$ by injectivity of I. Since this quantity also only depends on I up to isomorphism, for any $X \in A$-mod we get a well-defined integer $[d, I(X)]$, where $X \to I(X)$ is a minimal injective envelope.

Proposition 4.1. *Assume $\mathbb{K}$ has characteristic zero. Let $E = \mathrm{D}A \oplus E_{n+1} \oplus \cdots \oplus E_t$ be a cogenerating A-module, with indecomposable non-injective summands E_j, and B its cogenerator-tilted algebra. Let $d \in \mathbb{Z}^n_{\ge 0}$ be a dimension vector for A, and let $m = (m_{n+1},\dots,m_t) \in \mathbb{Z}^{n-t}_{\ge 0}$.*

(1) *If $\mathcal{C}^E_m \ne \varnothing$, then $[d, I(E_j)] \ge m_j$ for all j.*

(2) *In this case, we may extend d to a dimension vector $(d, s) \in \mathbb{Z}^t_{\geq 0}$ for B by defining $s_j := [d, I(E_j)] - m_j$ for $n + 1 \leq j \leq t$, and the special idempotent e of B induces an isomorphism*

$$R_B(d, s)/\!\!/\mathbf{Gl}_s \xrightarrow{\sim} \mathcal{C}^E_m.$$

Definition 4.2. Let M be a finite-dimensional A-module. We say M is *gen-finite* if there is a finite-dimensional A-module E such that $\operatorname{gen}(M) = \operatorname{add} E$.

The module A is gen-finite if and only if A is representation-finite. Thus we see gen-finiteness as a module-theoretic generalisation of the notion of representation-finiteness for algebras. For gen-finite modules, Zwara found the following explicit description of orbit closures as rank varieties.

Theorem 4.3 (cf. [6, Thm. 1.2(4)])**.** *Let $M \in R_A(d)$. If $\operatorname{gen}(M) = \operatorname{add} E$, then we have $\overline{\mathcal{O}}_M = \mathcal{C}^E_M$.*

The main step in our argument that the map π from Section 2 is a desingularisation is the following theorem, characterising the stable (d, s)-dimensional B-modules in $\overline{\mathcal{O}}_{c(M)}$ and giving a sufficient condition for them to be smooth points of this variety.

Theorem 4.4. *Let A and B be (any) basic algebras with $A \cong eBe$ for some idempotent e and let c be the intermediate extension functor associated to e. Let $\widetilde{N} \in R_B(d, s)$ and write $N = e\widetilde{N} \in R_A(d)$. Then the following are equivalent:*

(1) *$\widetilde{N} \in \overline{\mathcal{O}}^{\mathrm{st}}_{c(M)}$, and*
(2) *there is an exact sequence*

$$0 \longrightarrow N \longrightarrow M \oplus Z \xrightarrow{p} Z \longrightarrow 0$$

such that $\widetilde{N} \cong \ker c(p)$.

If condition (2) holds and we may choose the sequence so that $c(M \oplus Z)$ is rigid, then $\widetilde{N}$ is a smooth point of $\overline{\mathcal{O}}^{\mathrm{st}}_{c(M)}$.

We are now ready to describe our desingularisation for the orbit closure of a gen-finite module.

Theorem 4.5. *Assume $M \in A\text{-mod}$ is gen-finite. Let E be a basic cogenerator with $\operatorname{gen}(M) \subseteq \operatorname{add} E$, and let B be the cogenerator-tilted algebra of E with special idempotent e. For $c\colon A\text{-mod} \to B\text{-mod}$ the intermediate extension functor corresponding to e, write $(d, s) = \underline{\dim}\, c(M)$, and let $\pi\colon R_B(d, s)^{\mathrm{st}}/\mathbf{Gl}_s \to \operatorname{im} e$ be the projective map constructed in [3, §6.3]. Then the restriction*

$$\pi\colon \overline{\mathcal{O}}^{\mathrm{st}}_{c(M)}/\mathbf{Gl}_s \to \overline{\mathcal{O}}_M$$

is a desingularisation with connected fibres.

5. Desingularisation of quiver Grassmannians

The main question here is how to characterise smoothness for quiver Grassmannians. We have the following result (which was explained to us by Andrew Hubery).

Lemma 5.1. *Let B be a finite-dimensional basic algebra and X a finite-dimensional B-module with $\operatorname{id} X \le 1$ and $\operatorname{Ext}^1_B(X,X) = 0$. If d is such that $\operatorname{Gr}_B\binom{X}{d} \neq \varnothing$ and $\operatorname{Ext}^1_B(U, X/U) = 0$ for every $U \in \operatorname{Gr}_B\binom{X}{d}$, then $\operatorname{Gr}_B\binom{X}{d}$ is smooth and equidimensional.*

Then, we prove the following lemma.

Lemma 5.2. *Let B be a finite-dimensional basic algebra and $X \in B\text{-mod}$. Assume $C \in B\text{-mod}$ has the properties that $\operatorname{Ext}^i_B(C,C) = 0$ for all $i > 0$ and $X \in \operatorname{add} C$. Then if $U \in \operatorname{Gr}_B\binom{X}{d}$ fits into a short exact sequence*

$$0 \longrightarrow U \longrightarrow C_0 \longrightarrow C_1 \longrightarrow 0$$

with $C_i \in \operatorname{add} C$, we have $\operatorname{Ext}^i_B(U, X/U) = 0$ for all $i > 0$. In particular, if every $U \in \operatorname{Gr}_B\binom{X}{d}$ fits into such a sequence, then $\operatorname{Gr}_B\binom{X}{d}$ is smooth and equidimensional by Lemma 5.1.

Using the previous lemma, we can prove the theorem.

Theorem 5.3. *Let M be a gen-finite module and let E be the cogenerator given by the direct sum of all indecomposable modules in $\operatorname{gen}(M)$ together with any remaining indecomposable injectives. Let B be the cogenerator-tilted algebra of E with special idempotent e, let c be the intermediate extension associated to e, and let (d,s) be a dimension vector for B. Then if the Grassmannian $\operatorname{Gr}_B\binom{c(M)}{d,s}$ is non-empty, it is (scheme-theoretically) smooth and equidimensional.*

We now give the construction of our promised desingularisation for the quiver Grassmannian $\operatorname{Gr}_A\binom{M}{d}$ of a gen-finite A-module M. Since M is gen-finite, there is finite set of modules $N_1, \dots, N_t$ with $\operatorname{Gr}_A\binom{M}{d} = \bigcup_{i=1}^t \overline{\mathcal{E}}^{[N_i]}$. We fix such a set of modules N_i with the property that each $\overline{\mathcal{E}}^{[N_i]}$ is an irreducible component.

As in Theorem 5.3, let E be the cogenerator given by the direct sum of indecomposables in $\operatorname{gen}(M)$ together with any remaining indecomposable injectives. As usual, let B be the cogenerator-tilted algebra of E, with special idempotent e and associated intermediate extension c. Write $(d, s_i) = \underline{\dim}\, c(M) - \underline{\dim}\, c(N_i)$, and consider the projective map $\operatorname{Gr}_B\binom{c(M)}{d,s_i} \to \operatorname{Gr}_A\binom{M}{d}$ induced by e. It restricts to a projective map

$$p_i \colon \overline{\mathcal{E}}^{[c(N_i)]} \to \overline{\mathcal{E}}^{[N_i]}.$$

Since $\overline{\mathcal{E}}^{[N_i]}$ is an irreducible component of $\operatorname{Gr}_A\binom{M}{d}$, each $\mathcal{E}^{[c(N_i)]}$ contains a non-empty open subset, so $\overline{\mathcal{E}}^{[c(N_i)]}$ is also an irreducible component—since $\operatorname{Gr}_B\binom{c(M)}{d,s_i}$ is smooth by Theorem 5.3 it is even a connected component. Furthermore, p_i is an isomorphism over an open subset of $\mathcal{E}^{[N_i]}$, by dualising the argument of [3, Thm. 7.1(3)]. Combining the various maps p_i, we obtain

$$p = \bigsqcup_{i=1}^{t} p_i \colon \bigsqcup_{i=1}^{t} \overline{\mathcal{E}}^{[c(N_i)]} \to \operatorname{Gr}_A\tbinom{M}{d}.$$

By Theorem 5.3, the domain of this map is smooth. In summary, we have the following result.

Corollary 5.4. *For every gen-finite A-module M and every dimension vector d, the map p as constructed above is a desingularisation of* $\mathrm{Gr}_A\binom{M}{d}$.

6. Example for the n-subspace quiver

Let A be the path algebra of the n-subspace quiver:

$$\begin{tikzcd} 1 \arrow[drr] & 2 \arrow[dr] & \cdots & n-1 \arrow[dl] & n \arrow[dll] \\ & & 0 & & \end{tikzcd}$$

When treating an A-module X as a representation of this quiver, we denote by X_i the linear map carried by the arrow $i \to 0$.

For both of our examples, we consider the A-module

$$M = \mathrm{D}A \oplus S(0) = \Big(\bigoplus_{i=0}^{n} S(i) \Big) \oplus Q(0),$$

where $S(i)$ denotes the simple at a vertex i and $Q(i)$ its minimal injective envelope. It follows that M is gen-finite, and indeed we may compute $\mathrm{gen}(M) = \mathrm{add}\, M$.

Next, we calculate the orbit closure $\overline{\mathcal{O}}_M \subseteq R_A\left(\begin{smallmatrix}2&2&\cdots&2&2\\&&2&&\end{smallmatrix}\right)$. Since $\mathrm{gen}(M) = \mathrm{add}\, M$ it follows from Theorem 4.3 that

$$\begin{aligned}
\overline{\mathcal{O}}_M &= \{ N \in R_A\left(\begin{smallmatrix}2&2&\cdots&2&2\\&&2&&\end{smallmatrix}\right) \mid [N,Y] \geq [M,Y] \text{ for all } Y \in \mathrm{add}\, M \} \\
&= \{ N \in R_A\left(\begin{smallmatrix}2&2&\cdots&2&2\\&&2&&\end{smallmatrix}\right) \mid [N,S(0)] \geq [M,S(0)] = 1 \} \\
&\cong \Big\{ \left(\begin{smallmatrix} a_i & b_i \\ c_i & d_i \end{smallmatrix}\right)_{1 \leq i \leq n} \in \mathrm{Mat}_{2\times 2}(\mathbb{K}) \mid \mathrm{rk} \left(\begin{smallmatrix} a_1 & b_1 & a_2 & b_2 & \cdots & a_n & b_n \\ c_1 & d_1 & c_2 & d_2 & \cdots & c_n & d_n \end{smallmatrix}\right) \leq 1 \Big\} \\
&\cong V(X_i Y_j - X_j Y_i,\ i \neq j) \subseteq \mathrm{Spec}\, \mathbb{K}[X_1, \dots, X_{2n}, Y_1, \dots, Y_{2n}].
\end{aligned}$$

For the third step, note that maps $N_i \colon \mathbb{K}^2 \to \mathbb{K}^2$ for $1 \leq i \leq n$ determine a module N with $[N, S(0)] \geq 1$ if and only if there is a non-zero vector (x,y) such that $(x,y)(N_1, N_2, \dots, N_n) = 0$, this being equivalent to the rank inequality. Thus $\overline{\mathcal{O}}_M$ is a determinantal variety. We also have $\overline{\mathcal{O}}_M \cong \overline{\mathcal{O}}_{\widetilde{M}}$, where $\widetilde{M}$ is the representation

$$\mathbb{K}^{2n} \xrightarrow{\left(\begin{smallmatrix} 1&0&1&0&\cdots&1&0 \\ 0&0&0&0&\cdots&0&0 \end{smallmatrix}\right)} \mathbb{K}^2$$

of the A_2-quiver, and hence $\overline{\mathcal{O}}_M$ is a normal and Cohen–Macaulay variety.

Choosing bases, we identify M as the point of $R_A\left(\begin{smallmatrix}2&2&\cdots&2&2\\&&2&&\end{smallmatrix}\right)$ given by linear maps $M_i \colon \mathbb{K}^2 \to \mathbb{K}^2$ with $M_i = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ for all $1 \leq i \leq n$. Then, writing d for the dimension vector given by 1 at every vertex, we may describe the quiver Grassmannian $\mathrm{Gr}_A\binom{M}{d}$ as

$$\begin{aligned}
\mathrm{Gr}_A\tbinom{M}{d} &= \{ (L_0, L_1, L_2, \dots, L_n) \in \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \mid M_i(L_i) \subseteq L_0,\ 1 \leq i \leq n \} \\
&= \{ (L_0, [0:1], \dots, [0:1]) \mid L_0 \in \mathbb{P}^1 \} \cup \{ ([1:0], L_1, \dots, L_n) \mid L_i \in \mathbb{P}^1,\ 1 \leq i \leq n \}.
\end{aligned}$$

Let $f \colon (\mathbb{P}^1)^n \to \mathrm{Gr}_A\binom{M}{d}$ be the regular map

$$f(t_1, \dots, t_n) = ([1:0], t_1, \dots, t_n)$$

and $g\colon \mathbb{P}^1 \to \mathrm{Gr}_A\binom{M}{d}$ the regular map

$$g(t) = (t, [0:1], \dots, [0:1]).$$

The images of these maps are closed and irreducible, cover $\mathrm{Gr}_A\binom{M}{d}$, and neither is contained in the other—therefore they are the irreducible components. If $U \in \operatorname{im} f$, then we calculate directly that $M/U \cong S$, so $\operatorname{im} f = \mathcal{E}^{[S]} = \overline{\mathcal{E}}^{[S]}$ is one irreducible component, isomorphic to $(\mathbb{P}^1)^n$. Similarly, $M/g(t) \cong Q$ for $t \neq [1:0]$, whereas $M/g([1:0]) \cong S$. Thus the other irreducible component is $\operatorname{im} g = \mathcal{E}^{[Q]} \sqcup \{U_0\} = \overline{\mathcal{E}}^{[Q]} \cong \mathbb{P}^1$ for $U_0 = g([1:0])$, this being the unique intersection point of $\overline{\mathcal{E}}^{[S]}$ and $\overline{\mathcal{E}}^{[Q]}$. In particular, U_0 is the only singular point of $\mathrm{Gr}_A\binom{M}{d}$.

6.1. **The cogenerator-tilted algebra.** For our constructions, we choose $E = M$, noting that M is a cogenerator, and add $M = \operatorname{gen}(M)$. Then $\Gamma = \operatorname{End}_A(E)^{\mathrm{op}} \cong \mathbb{K}Q_\Gamma/\operatorname{rad}^2(Q_\Gamma)$ for Q_Γ the quiver

$$\begin{tikzcd}
1' \arrow[drr] & 2' \arrow[dr] & \cdots & (n-1)' \arrow[dl] & n' \arrow[dll] \\
 & & 0' \arrow[d] & & \\
 & & \blacklozenge & &
\end{tikzcd}$$

Here each vertex i' corresponds to the summand $Q(i)$ of M, noting that $Q(i) = S(i)$ for $i \geq 1$, and $\blacklozenge$ corresponds to $S(0)$.

The projective Γ-module $P = \operatorname{Hom}_A(E, \mathrm{D}A) = \bigoplus_{i=0}^{n} P(i')$ is faithful. Since $[S(\blacklozenge), P(i')] = 0$ for $1 \leq i \leq n$, a minimal left $\operatorname{add} P$-approximation of $S(\blacklozenge)$ is given by a monomorphism $S(\blacklozenge) \to P(0')$, with cokernel $S(0')$. This implies $T_P = P \oplus S(0')$ is the P-special tilting Γ-module. Then we calculate that the cogenerator-tilted algebra $B = \operatorname{End}_\Gamma(T_P)^{\mathrm{op}}$ of E is isomorphic to the path algebra $\mathbb{K}Q_B$ for Q_B the quiver

$$\begin{tikzcd}
{[1]} \arrow[drr] & {[2]} \arrow[dr] & \cdots & {[n-1]} \arrow[dl] & {[n]} \arrow[dll] \\
 & & \lozenge \arrow[d] & & \\
 & & {[0]} & &
\end{tikzcd}$$

The vertex $[i]$ corresponds to the summand $P(i')$ of T_P, and $\lozenge$ to the summand $S(0') = \Omega^{-1}S(\blacklozenge)$. The special idempotent is $e := \sum_{i=0}^{n} e_{[i]}$, corresponding to the summand P of T_P, and we can check that $eBe \cong A$ as expected. Set $C := c(M)$, where c is the intermediate extension corresponding to e. Since c maps simples to simples [5, §4] and injectives to injectives we may calculate

$$c(M) = c(S) \oplus c(Q) = \Big(\bigoplus_{i=0}^{n} S[i]\Big) \oplus Q[0],$$

and so $\underline{\dim}\, c(M) = \left(\begin{smallmatrix} 2\ 2\ \cdots\ 2\ 2 \\ 1 \\ 2 \end{smallmatrix}\right)$.

6.2. **Desingularisation of the orbit closure.** To desingularise $\overline{\mathcal{O}}_M$, we are interested in the stable B-modules, which are the modules in $\operatorname{cogen}(\widetilde{Q})$ for $\widetilde{Q} = \mathrm{D}(eB) = \bigoplus_{i=1}^n S[i] \oplus Q[0]$. These are the modules with socle supported away from $\Diamond$, or equivalently, in the language of quiver representations, those for which the arrow $\Diamond \to [0]$ carries a monomorphism. It follows that

$$X := R_B\left(\begin{smallmatrix} 2\ 2\ \cdots\ 2\ 2 \\ 1 \\ 2 \end{smallmatrix}\right)^{\mathrm{st}}/\mathbf{Gl}_1 = \{(N, U) \in R_A(d) \times \mathbb{P}^1 \mid \operatorname{im} N_i \subseteq U,\ 1 \leq i \leq n\},$$

where, under this identification, U is the image of the monomorphism on the arrow $\Diamond \to [0]$.

We can check that X is smooth and irreducible by considering the projection $\mathrm{pr}_2 \colon X \to \mathbb{P}^1$. The fibre over $[1:0]$ consists of all tuples $(N_1, \dots, N_n) \in \mathrm{Mat}_{2\times 2}(\mathbb{K})^n$ such that each N_i has lower row zero, and hence this fibre is an affine space (of dimension $2n$). In fact, it is a $\mathbb{B}$-representation where $\mathbb{B} \subseteq \mathbf{Gl}_2(\mathbb{K})$ denotes the upper triangular matrices operating by conjugation. Since pr_2 is a $\mathbf{Gl}_2$-equivariant map into the homogeneous space $\mathbb{P}^1$, it follows that X is is a vector bundle over $\mathbb{P}^1$, and so is smooth and irreducible. In particular, since $c(M)$ is rigid, $X = \overline{\mathcal{O}}^{\mathrm{st}}_{c(M)}$. Thus the desingularisation of $\overline{\mathcal{O}}_M$ from Theorem 4.5 is

$$\pi = \mathrm{pr}_1 \colon \{(N, U) \in R_A(d) \times \mathbb{P}^1 \mid \operatorname{im} N_i \subseteq U,\ 1 \leq i \leq n\} \to \overline{\mathcal{O}}_M.$$

6.3. **Desingularisation of the quiver Grassmannian.** Now we describe our desingularisation of $\mathrm{Gr}_A\binom{M}{d}$. Let

$$d_Q := \underline{\dim}\, c(M) - \underline{\dim}\, c(Q) = \left(\begin{smallmatrix} 1\ 1\ \cdots\ 1\ 1 \\ 0 \\ 1 \end{smallmatrix}\right)$$

and

$$d_S := \underline{\dim}\, c(M) - \underline{\dim}\, c(S) = \left(\begin{smallmatrix} 1\ 1\ \cdots\ 1\ 1 \\ 1 \\ 1 \end{smallmatrix}\right).$$

Every module of dimension vector d_Q is isomorphic to $c(S)$, so the Grassmannian $\mathrm{Gr}_B\binom{c(M)}{d_Q} = \mathcal{S}_{[c(S)]}$ is smooth and irreducible of dimension $[c(S), c(M)] - [c(S), c(S)] = [c(S), c(Q)] = [S, Q] = 1$, and hence it coincides with $\overline{\mathcal{E}}^{[c(Q)]}$. Similarly, by considering the quotients, $\mathrm{Gr}_B\binom{c(M)}{d_S} = \mathcal{E}^{[c(S)]}$ is smooth and irreducible of dimension $[c(M), c(S)] - [c(S), c(S)] = [Q, S] = n$, and so in particular $\overline{\mathcal{E}}^{[c(S)]} = \mathrm{Gr}_B\binom{c(M)}{d_S}$. Thus the theoretical desingularisation given by Corollary 5.4 coincides with the naïve desingularisation $p \colon \mathrm{Gr}_B\binom{c(M)}{d_Q} \sqcup \mathrm{Gr}_B\binom{c(M)}{d_S} \to \mathrm{Gr}_A\binom{M}{d}$, given by taking the disjoint union of the two irreducible components.

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