STRONGLY QUASI-HEREDITARY ALGEBRAS AND REJECTIVE SUBCATEGORIES

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Abstract. Ringel's right-strongly quasi-hereditary algebras are a special class of quasi-hereditary algebras of Cline-Parshall-Scott. We give characterizations of these algebras in terms of heredity chains and right rejective subcategories. As applications, we prove that any artin algebra of global dimension at most two is right-strongly quasi-hereditary. Moreover we show that the Auslander algebra of a representation-finite algebra A is strongly quasi-hereditary if and only if A is a Nakayama algebra.

1. Strongly quasi-hereditary algebras

Throughout this note, A is an artin algebra and $J(A)$ is the Jacobson radical of A. We denote by $\mathsf{mod}A$ the category of finitely generated right A-modules and by $\mathsf{proj}A$ the category of finitely generated projective right A-modules. For $M \in \mathsf{mod}A$, we write $\mathsf{add}M$ for the category of all direct summands of finite direct sums of copies of M.

We fix a complete set of representatives of isomorphism classes of simple A-modules $\{S(i) \mid i \in I\}$. For $i \in I$, we denote by $P(i)$ the projective cover of $S(i)$. Let $\leq$ be a partial order on I. For each $i \in I$, we denote by $\Delta(i)$ the maximal factor module of $P(i)$ whose composition factors have the form $S(j)$ for some $j \leq i$. The module $\Delta(i)$ is called the *standard module* corresponding to i. Let $\Delta := \{\Delta(i) \mid i \in I\}$ be the set of standard modules. We denote by $\mathcal{F}(\Delta)$ the full subcategory of $\mathsf{mod}A$ whose objects are the modules which have a Δ-filtration.

We recall the definition of quasi-hereditary algebras. A two-sided ideal H of A is called an idempotent ideal if $H^2 = H$, or equivalently, there exists an idempotent e such that $H = AeA$.

Definition 1 (Cline-Parshall-Scott [1], Dlab-Ringel [3])**.** An artin algebra A is called a *quasi-hereditary* algebra if A admits a heredity chain, *i.e.*, there exists a chain of idempotent ideals

$$0 = H_n < H_{n-1} < \cdots < H_{i+1} < H_i < \cdots < H_0 = A$$

such that $H_i/H_{i+1} \in \mathsf{proj}A/H_{i+1}$ and $H_i/H_{i+1}J(A/H_{i+1})H_i/H_{i+1} = 0$ for all $i \in I$.

Quasi-hereditary algebras are strongly related to highest weight categories defined below. In fact, an artin algebra A is quasi-hereditary if and only if there exists a partial order $\leq$ on I such that a pair $(\mathsf{mod}A, \leq)$ is a highest weight category [1, Theorem 3.6].

A pair $(\mathsf{mod}A, \leq)$ is called a highest weight category if there exists a short exact sequence

$$0 \to K(i) \to P(i) \to \Delta(i) \to 0$$

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for any $i \in I$ with the following properties:

(a) $K(i) \in \mathcal{F}(\Delta)$ for all $i \in I$;
(b) if $(K(i) : \Delta(j)) \neq 0$, then we have $i < j$.

Motivated by Iyama's finiteness theorem of representation dimensions of artin algebras ([4], [5]), Ringel introduced the notion of right-strongly quasi-hereditary algebras from the viewpoint of highest weight categories.

Definition 2 (Ringel [6, §4])**.** A pair $(A, \leq)$ (or simply A) is called a *right-strongly quasi-hereditary* algebra if a pair $(\mathsf{mod}A, \leq)$ is a highest weight category such that each standard module has projective dimension at most one. Dually, we define *left-strongly quasi-hereditary* algebras.

We start with the following observation which gives a characterization of right-strongly (resp. left-strongly) quasi-hereditary algebras in terms of heredity chains.

Proposition 3. [7, Proposition 3.7] *Let A be an artin algebra. Then A is right-strongly* (*resp. left-strongly*) *quasi-hereditary if and only if there exists a chain of idempotent ideals*

$$0 = H_n < H_{n-1} < \cdots < H_{i+1} < H_i < \cdots < H_0 = A$$

such that H_i is a projective right (*resp. left*) *A-module and $H_i/H_{i+1}J(A/H_{i+1})H_i/H_{i+1} = 0$ for any $i \in I$.*

We call such a chain a *right-strongly* (resp. *left-strongly*) *heredity chain*. Note that all right-strongly (resp. left-strongly) heredity chains are heredity chains.

We introduce a special class of right-strongly quasi-hereditary algebras.

Definition 4. An artin algebra is said to be *strongly quasi-hereditary* if it has a right-strongly heredity chain such that it is a left-strongly heredity chain.

2. Main result

We give categorical interpretations of right-strongly (resp. left-strongly) heredity chains by using right (resp. left) rejective subcategories. We start with recalling the definitions of right rejective subcategories and coreflective subcategories. In the following, by a subcategory, we mean a full subcategory which is closed under isomorphisms, direct sums and direct summands.

Definition 5 (Freyd, Iyama [4, 2.1(1)])**.** Let $\mathcal{C}$ be an additive category and $\mathcal{C}'$ a subcategory of $\mathcal{C}$.

(1) We call $\mathcal{C}'$ a *coreflective* (resp. *reflective*) subcategory of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a right (resp. left) adjoint.
(2) We call $\mathcal{C}'$ a *right* (resp. *left*) *rejective* subcategory of $\mathcal{C}$ if the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ has a right (resp. left) adjoint with a counit ε^- (resp. unit ε^+) such that ε^-_X is a monomorphism (resp. ε^+_X is an epimorphism) for $X \in \mathcal{C}$.
(3) We call $\mathcal{C}'$ a *rejective subcategory* of $\mathcal{C}$ if $\mathcal{C}'$ is a right and left rejective subcategory of $\mathcal{C}$.

We give a typical example of a rejective subcategory. Let B be an arbitrary factor algebra of A. We naturally regard $\mathsf{mod}B$ as a full subcategory of $\mathsf{mod}A$. Then $\mathsf{mod}B$ is a rejective subcategory of $\mathsf{mod}A$.

We introduce the central notion of this note by using right rejective subcategories and coreflective subcategories.

Definition 6 (Iyama [4], [7])**.** Let $\mathcal{C}$ be a Krull-Schmidt category. A chain of subcategories of $\mathcal{C}$

$$0 = \mathcal{C}_n \subset \mathcal{C}_{n-1} \subset \cdots \subset \mathcal{C}_i \subset \cdots \subset \mathcal{C}_0 = \mathcal{C} \tag{2.1}$$

is called a *total right rejective* (resp. *coreflective*) *chain* if $\mathcal{C}_i$ is a right rejective (resp. coreflective) subcategory of $\mathcal{C}$ and the quotient category $\mathcal{C}_i/[\mathcal{C}_{i+1}]$ is semisimple for all i. Dually, we define *total left rejective chains* and *reflective chains*. Moreover we call (2.1) a *rejective chain* if (2.1) is a total right rejective chain and a total left rejective chain.

The following main theorem in this note characterizes right-strongly (resp. left-strongly) quasi-hereditary algebras in terms of these chains.

Theorem 7. [7, Theorem 1.2] *Let A be an artin algebra and*

$$0 = H_n < H_{n-1} < \cdots < H_{i+1} < H_i < \cdots < H_0 = A \tag{2.2}$$

a chain of idempotent ideals of A. For $0 \leq i \leq n-1$, we write $H_i = Ae_iA$, where e_i is an idempotent of A. Then the following conditions are equivalent.

(i) (2.2) *is a right-strongly* (*resp. left-strongly*) *heredity chain.*
(ii) *The following chain is a total right* (*resp. left*) *rejective chain of* $\mathsf{proj}A$*.*

$$0 = \mathsf{add}e_nA \subset \mathsf{add}e_{n-1}A \subset \cdots \subset \mathsf{add}e_iA \subset \cdots \subset \mathsf{add}e_0A = \mathsf{proj}A.$$

(iii) (2.2) *is a heredity chain of A and the following chain is a coreflective* (*resp. reflective*) *chain of* $\mathsf{proj}A$*.*

$$0 = \mathsf{add}e_nA \subset \mathsf{add}e_{n-1}A \subset \cdots \subset \mathsf{add}e_iA \subset \cdots \subset \mathsf{add}e_0A = \mathsf{proj}A.$$

In particular, A is a strongly quasi-hereditary algebra if and only if $\mathsf{proj}A$ *has a rejective chain.*

We give a sketch of the proof of Theorem 7. The chain (2.2) induces a chain of full subcategories of $\mathsf{proj}A$

$$0 = \mathsf{add}e_nA \subset \cdots \subset \mathsf{add}e_0A = \mathsf{proj}A.$$

Then the assertion follows from the following lemma.

Lemma 8. *Let A be an artin algebra. Then the following statements hold.*

(1) (Iyama [5, Theorem 3.2 (2)]) An idempotent ideal AeA is a projective right A-module if and only if $\mathsf{add}eA$ is a right rejective subcategory of $\mathsf{proj}A$.
(2) Let $Ae'A \subset AeA$ be idempotent ideals of A. Then the following conditions are equivalent.
  (i) The quotient category $\mathsf{add}eA/[\mathsf{add}e'A]$ is a semisimple category.
  (ii) $(AeA/Ae'A)J(A/Ae'A)(AeA/Ae'A) = 0$.

One of the advantages of Theorem 7 is that we can give a right-strongly (resp. left-strongly, strongly) structure without a quasi-hereditary structure. We apply Theorem 7 to the following well-known result.

Example 9. (1) (Iyama [4, Theorem 1.1], Ringel [6, Theorem in §5]) Let A be an artin algebra. Then there exists a right-strongly quasi-hereditary algebra B and an idempotent e of B such that $A = eBe$.
(2) (Conde [2]) Auslander-Dlab-Ringel algebras are left-strongly quasi-hereditary algebras.

3. Applications

In the rest of this note, we give applications of Theorem 7. First we sharpen a well-known result of Dlab-Ringel [3, Theorem 2] stating that any artin algebra of global dimension at most two is quasi-hereditary. We prove that such an algebra is always right-strongly (resp. left-strongly) quasi-hereditary.

Theorem 10. [7, Theorem 4.1] *Let A be an artin algebra. If the global dimension of A is at most two, then A is a right-strongly quasi-hereditary algebra.*

Remark 11. We obtain that, if the global dimension of A is at most two, then A is a left-strongly quasi-hereditary algebra by Theorem 7. Hence any artin algebra of global dimension at most two is right-strongly quasi-hereditary and left-strongly quasi-hereditary. However it is not necessarily strongly quasi-hereditary.

By Theorem 10, Auslander algebras are right-strongly quasi-hereditary. However they are not necessarily strongly quasi-hereditary. Applying Theorem 7, we obtain the following characterization of Auslander algebras to be strongly quasi-hereditary.

Theorem 12. [7, Theorem 4.6] *Let A be a representation-finite artin algebra and B the Auslander algebra of A. Then the following statements are equivalent.*
(i) *A is a Nakayama algebra.*
(ii) *B is a strongly quasi-hereditary algebra.*

A key of the proof of Theorem 12 is that B is strongly quasi-hereditary if and only if $\mathsf{proj}B$ has a rejective chain by Theorem 7. Since $\mathsf{proj}B \simeq \mathsf{mod}A$, we show that $\mathsf{mod}A$ has a rejective chain if and only if A is a Nakayama algebra.

References

[1] E. Cline, B. Parshall, and L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99. MR 961165
[2] T. Conde, *The quasihereditary structure of the Auslander-Dlab-Ringel algebra*, J. Algebra **460** (2016), 181–202. MR 3510398
[3] V. Dlab and C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. **33** (1989), no. 2, 280–291. MR 987824
[4] O. Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011–1014 (electronic). MR 1948089
[5] ———, *Rejective subcategories of artin algebras and orders*, arXiv preprint, nov 2003.
[6] C. M. Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692. MR 2593693

[7] M. Tsukamoto, *Strongly quasi-hereditary algebras and rejective subcategories*, arXiv preprints, jun 2017.

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