WIDE SUBCATEGORIES ARE SEMISTABLE

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Abstract. In this note, we give that for an arbitrary finite dimensional algebra Λ, any wide subcategory of $\mathsf{mod}\Lambda$ satisfying a certain finiteness condition is θ-semistable for some stability condition θ. More generally, we give that wide subcategories of $\mathsf{mod}\Lambda$ associated with two-term presilting complexes of Λ are semistable. This provides a complement for Ingalls-Thomas-type bijections for finite dimensional algebras.

1. Wide/Semistable subcategories

This is a report on results presented in [8]. Throughout this note, Λ is a finite dimensional algebra over a field k. Let $\mathsf{mod}\Lambda$ (resp., $\mathsf{proj}\Lambda$) be the category of finitely generated right Λ-modules (resp., projective right Λ-modules).

Wide subcategories of $\mathsf{mod}\Lambda$ are full subcategories closed under kernels, cokernels and extensions. Important examples of wide subcategories are given by geometric invariant theory for quiver representations [5]. Recall that a *stability condition* on $\mathsf{mod}\Lambda$ is a linear form θ on $K_0(\mathsf{mod}\Lambda) \otimes_{\mathbb{Z}} \mathbb{R}$, where $K_0(\mathsf{mod}\Lambda)$ is the Grothendieck group of $\mathsf{mod}\Lambda$. We say that $M \in \mathsf{mod}\Lambda$ is *θ-semistable* if $\theta(M) = 0$ and $\theta(L) \leq 0$ for any submodule L of M, or equivalently, $\theta(N) \geq 0$ for any factor module N of M. The full subcategory of θ-semistable Λ-modules is called the *θ-semistable subcategory* of $\mathsf{mod}\Lambda$. It is basic that semistable subcategories of $\mathsf{mod}\Lambda$ are wide.

In this note, we give that any wide subcategory of $\mathsf{mod}\Lambda$ satisfying a certain finiteness condition is θ-semistable for some stability condition θ. Our original motivation comes from bijections given by Ingalls and Thomas [3].

2. Ingalls-Thomas bijections

We recall basic notations and definitions. Let $\mathcal{S}$ be a full subcategory of $\mathsf{mod}\Lambda$. For a full subcategory $\mathcal{S}$ of $\mathsf{mod}\Lambda$, let

$$\mathcal{S}^{\perp} := \{M \in \mathsf{mod}\Lambda \mid \mathrm{Hom}_{\Lambda}(\mathcal{S}, M) = 0\}, \quad {}^{\perp}\mathcal{S} := \{M \in \mathsf{mod}\Lambda \mid \mathrm{Hom}_{\Lambda}(M, \mathcal{S}) = 0\}.$$

We call $\mathcal{S}$ a *torsion class* (resp., *torsion free class*) if it is closed under extensions and quotients (resp., extensions and submodules). For subcategories $\mathcal{T}$ and $\mathcal{F}$ of $\mathsf{mod}\Lambda$, a pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* if $\mathcal{T} = {}^{\perp}\mathcal{F}$ and $\mathcal{F} = \mathcal{T}^{\perp}$. Then $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion free class. Conversely, any torsion class (resp., torsion free class) gives rise to a torsion pair. We call $\mathcal{S}$ *functorially finite* if any Λ-module admits both a left and a right $\mathcal{S}$-approximation. We call $\mathcal{S}$ *left finite* if the minimal torsion class containing $\mathcal{S}$ is functorially finite.

The detailed version of this paper will be submitted for publication elsewhere.

For quiver representations, Ingalls and Thomas [3] gave bijections between wide/semistable subcategories and other important objects.

Theorem 1. [3] *For the path algebra kQ of a finite connected acyclic quiver Q over a field k, there are bijections between the following objects, where we refer to* [3] *for the notion of* support tilting *modules.*

1. *Isomorphism classes of basic support tilting modules in* $\mathsf{mod}(kQ)$.
2. *Functorially finite torsion classes in* $\mathsf{mod}(kQ)$.
3. *Functorially finite wide subcategories of* $\mathsf{mod}(kQ)$.
4. *Functorially finite semistable subcategories of* $\mathsf{mod}(kQ)$.

Later, works of Adachi-Iyama-Reiten [1] and Marks-Stovicek [6] gave a generalization of Theorem 1 (called *Ingalls-Thomas-type bijections*) for an arbitrary finite dimensional k-algebra.

3. Ingalls-Thomas-type bijections

For an additive (resp., abelian) category $\mathcal{A}$, let $\mathsf{K}^{\mathrm{b}}(\mathcal{A})$ (resp., $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$) be the homotopy (resp., derived) category of bounded complexes over $\mathcal{A}$. Let $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$. We call P *presilting* if $\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)}(P, P[i]) = 0$ for any $i > 0$. We call P *silting* if P is presilting and satisfies $\mathsf{thick}P = \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$, where $\mathsf{thick}P$ is the smallest subcategory of $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ containing P which is closed under shifts, cones and direct summands. We say that $P = (P^i, d^i)$ is *two-term* if $P^i = 0$ for all $i \neq 0, -1$. We denote by 2-presiltΛ (resp., 2-siltΛ) the set of isomorphism classes of basic two-term presilting (resp., silting) complexes in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.

For $M \in \mathsf{mod}\Lambda$, let $\mathsf{add}M$ (resp., $\mathsf{Fac}M$, $\mathsf{Sub}M$) be the category of all direct summands (resp., factor modules, submodules) of finite direct sums of copies of M.

We ready to note Ingalls-Thomas-type bijections.

Theorem 2. [1, 6] *There are bijections between the following objects, where we refer to* [1] *for the notion of* support τ-tilting *modules.*

- (1) *Isomorphism classes of basic support τ-tilting modules in* $\mathsf{mod}\Lambda$.
- (1′) *Isomorphism classes of basic two-term silting complexes in* $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$.
- (2) *Functorially finite torsion classes in* $\mathsf{mod}\Lambda$.
- (2′) *Functorially finite torsion free classes in* $\mathsf{mod}\Lambda$.
- (3) *Left finite wide subcategories of* $\mathsf{mod}\Lambda$.

Theorem 2 are given in the following way [1, 6, 7]:

$$\begin{array}{lll}
(1') \to (1) & : & T \mapsto \mathrm{H}^0(T), \\
(1') \to (2) & : & T \mapsto \mathsf{Fac}\mathrm{H}^0(T), \quad (1') \to (2') \ : \ T \mapsto \mathsf{Sub}\mathrm{H}^{-1}(\nu T), \\
(2) \to (2') & : & \mathcal{T} \mapsto \mathcal{T}^{\perp}, \qquad\qquad (2) \leftarrow (2') \ : \ {}^{\perp}\mathcal{F} \leftarrow\!\shortmid \mathcal{F}, \\
(1') \to (3) & : & T \mapsto \mathcal{W}^T := \mathsf{Fac}\mathrm{H}^0(T_\lambda) \cap \mathrm{H}^0(T_\rho)^{\perp},
\end{array}$$

where $\mathrm{H}^i(T)$ is the i-th cohomology of T and ν is the Nakayama functor.

Notice that the statement for semistable subcategories of $\mathsf{mod}\Lambda$ is missing in Theorem 2. Our aim is to provide a complement for Theorem 2.

4. Our results

The following main theorem provides a complement for Ingalls-Thomas-type bijections.

Theorem 3. [8] *The following objects are the same.*

(3) *Left finite wide subcategories of* $\mathsf{mod}\Lambda$.

(4) *Left finite semistable subcategories of* $\mathsf{mod}\Lambda$.

Therefore, there are bijections between $(1)-(3)$ *in Theorem 2 and* (4).

Since it is basic that semistable subcategories are wide subcategories, it suffices to show the converse. To construct a stability condition θ for a given left finite wide subcategory, we need the following preparation. There are natural isomorphisms $K_0(\mathsf{mod}\Lambda) \simeq K_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda))$ and $K_0(\mathsf{proj}\Lambda) \simeq K_0(\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda))$. Moreover, $K_0(\mathsf{mod}\Lambda)$ has a basis consisting of the isomorphism classes S_i of simple Λ-modules, and $K_0(\mathsf{proj}\Lambda)$ has a basis consisting of the isomorphism classes P_i of indecomposable projective Λ-modules, where $\mathsf{top}P_i = S_i$. The Euler form is a non-degenerate pairing between $K_0(\mathsf{proj}\Lambda)$ and $K_0(\mathsf{mod}\Lambda)$ given by

$$\langle P, X\rangle := \sum_{i\in\mathbb{Z}} (-1)^i \dim_k \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)}(P, X[i])$$

for any $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$ and $X \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\Lambda)$. Then $\{P_i\}$ and $\{S_i\}$ satisfies $\langle P_i, S_j\rangle = \delta_{ij}\dim_k \mathrm{End}_\Lambda(S_j)$ for any i and j, where δ_{ij} is the Kronecker delta. In particular, we have a $\mathbb{Z}$-linear form $\langle P, -\rangle : K_0(\mathsf{mod}\Lambda) \to \mathbb{Z}$ for $P \in \mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$. On the other hand, for $T \in 2\text{-silt}\Lambda$, there is a decomposition $T = T_\lambda \oplus T_\rho$ and a triangle

$$\Lambda \to T' \to T'' \to \Lambda[1]$$

in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\Lambda)$, where $\mathsf{add}T' = \mathsf{add}T_\lambda$ and $\mathsf{add}T'' = \mathsf{add}T_\rho$ (see [2]).

Our Theorem 3 is a consequence of the following result.

Theorem 4. [8] *For* $T \in 2\text{-silt}\Lambda$, *we consider an* $\mathbb{R}$*-linear form* θ *defined by*

$$\sum_X a_X \langle X, -\rangle : K_0(\mathsf{mod}\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

where X *runs over all indecomposable direct summands of* T_ρ, *and* a_X *is an arbitrary positive real number for each* X. *Then* $\mathcal{W}^T$ *is the* θ*-semistable subcategory of* $\mathsf{mod}\Lambda$.

We give Theorem 4 in a more general setting. Any $U \in 2\text{-presilt}\Lambda$ gives rise to a wide subcategory of $\mathsf{mod}\Lambda$ as follows: By [1], there are two torsion pairs

$$({}^\perp\mathrm{H}^{-1}(\nu U), \mathsf{Sub}\mathrm{H}^{-1}(\nu U)), \quad (\mathsf{Fac}\mathrm{H}^0(U), \mathrm{H}^0(U)^\perp)$$

in $\mathsf{mod}\Lambda$ such that ${}^\perp\mathrm{H}^{-1}(\nu U) \supseteq \mathsf{Fac}\mathrm{H}^0(U)$ and $\mathsf{Sub}\mathrm{H}^{-1}(\nu U) \subseteq \mathrm{H}^0(U)^\perp$. Note that the equalities hold if and only if $U \in 2\text{-silt}\Lambda$. Then it is easy to show that

$$\mathcal{W}_U := {}^\perp\mathrm{H}^{-1}(\nu U) \cap \mathrm{H}^0(U)^\perp$$

is a wide subcategory of $\mathsf{mod}\Lambda$, which is equivalent to $\mathsf{mod}C$ for some explicitly constructed finite dimensional algebra C (see [4]).

Lemma 5. [8] *Let* $T = T_\lambda \oplus T_\rho \in 2\text{-silt}\Lambda$. *Then* $\mathcal{W}^T = \mathcal{W}_{T_\rho}$ *holds.*

Our Theorem 4 is a consequence of Lemma 5 and the following result.

Theorem 6. [8] *For $U \in$ 2-presiltΛ, we consider an $\mathbb{R}$-linear form θ defined by*

$$\sum_X a_X \langle X, - \rangle : K_0(\mathsf{mod}\Lambda) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R},$$

where X runs over all indecomposable direct summands of U, and a_X is an arbitrary positive real number for each X. Then $\mathcal{W}_U$ is the θ-semistable subcategory of $\mathsf{mod}\Lambda$.

References

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