The 51st Symposium on Ring Theory

and Representation Theory

ABSTRACT

Okayama University of Science, Okayama

September 19 – 22, 2018

Program

September 19 (Wednesday)

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Frobenius ring homomorphisms

10:20–10:50 Yoshihiro Otokita (Chiba University)

Centers of modular group algebras

11:00–12:00 Kohji Yanagawa (Kansai University)

Homological methods in combinatorial commutative algebra I

13:10–13:40 Yusuke Nakajima (The University of Tokyo)

Finite dimensional algebras arising from dimer models and their derived equivalences

13:50–14:20 Toshitaka Aoki (Nagoya University)

Torsion classes for algebras with radical square zero

14:30–15:00 Norihiro Hanihara (Nagoya University)

Auslander correspondence for triangulated categories

15:20–15:50 Sota Asai (Nagoya University)

The chamber structures of the Grothendieck groups coming from bricks

16:00–17:00 Yu Qiu (Tsinghua University)

Decorated marked surfaces

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Connectedness of the Balmer spectrum of the right bounded derived category of a commutative noetherian ring

10:20–10:50 Tokuji Araya (Okayama University of Science), Kei-ichiro Iima (Nara College), Maiko Ono (Okayama University), Ryo Takahashi (Nagoya University)

Dimensions of singular categories of hypersurfaces of countable representation type

11:00–12:00 Kohji Yanagawa (Kansai University)

Homological methods in combinatorial commutative algebra II

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Resolution of DG-modules and their applications for commutative DG-algebras

13:50–14:20 Haruhisa Enomoto (Nagoya University)

Relations for Grothendieck groups and representation-finiteness

14:30–15:00 Yuya Mizuno (Shizuoka University)

Sortable elements and torsion pairs for quivers

15:20–15:50 Takahide Adachi (Osaka Prefecture University), Aaron Chan (Nagoya University)

A geometric model of Brauer graph algebras

16:00–16:30 Kaoru Motose
Cyclotomic polynomials

17:30– Conference dinner

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Batalin-Vilkovisky algebra structures on the Hochschild cohomology of self-injective Nakayama algebras

10:20–10:50 Kazunori Nakamoto (University of Yamanashi), Takeshi Torii (Okayama University)
An application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring

11:00–12:00 Hiroyuki Nakaoka (Kagoshima University)
Some remarks on Avella-Alaminos-Geiss invariants of gentle algebras

13:10–13:40 Masahisa Sato (Aichi University & Yamanashi University)
On projective modules with unique maximal submodule

13:50–14:20 Gangyong Lee, Mauricio Medina-Bárcenas (Chungnam National University)
On (finite) Σ-Rickart modules: On a module theoretic setting of the (semi-)hereditary property of rings

14:30–15:00 Mayu Tsukamoto (Osaka City University)
A strongly quasi-hereditary structure on Auslander–Dlab–Ringel algebras

15:20–15:50 Kengo Miyamoto (Osaka University)
Components of the stable Auslander-Reiten quiver for a symmetric order over a complete discrete valuation ring

16:00–17:00 Yu Qiu (Tsinghua University)
$\mathbb{X}$-stability conditions on Calabi-Yau-$\mathbb{X}$ categories

September 22 (Saturday)

9:40–10:10 Shigeo Koshitani, Taro Sakurai (Chiba University)
On certain Morita invariants involving commutator subspace and radical powers

10:20–10:50 Ayako Itaba (Tokyo University of Science), Masaki Matsuno (Shizuoka University)
The defining relations of geometric algebras of Type EC

11:00–11:30 Akira Masuoka (University of Tsukuba)
Quotients G/H is super-symmetry

Frobenius ring homomorphisms

Mitsuo Hoshino, Noritsugu Kameyama and Hirotaka Koga

Let R be a left and right noetherian ring. We call a ring homomorphism $\phi : R \to A$ Frobenius if A is a finitely generated right R-module with $\mathrm{Ext}^i_R(A, R) = 0$ for all $i \geq 1$ and if $\mathrm{Hom}_R(A, R)$ is a finitely generated projective generator for right A-modules (cf. [1, 7, 8]). Our aim is to show that if a Frobenius ring homomorphism $\phi : R \to A$ is given then A inherits various homological properties from R.

Assume that a Frobenius ring homomorphism $\phi : R \to A$ is given. Then A also is left and right noetherian (cf. [5]). Furthermore, if R is Auslander-Gorenstein then so is A with inj dim $A \leq$ inj dim R (cf. [3, 4, 6, 9]). Also, in connection with the generalized Nakayama conjecture ([2]), we notice that $\mathrm{Ext}^i_A(X, R) = 0$ if and only if $\mathrm{Ext}^i_R(X, R) = 0$ for any right A-module X and $i \geq 0$, and that a ring homomorphism $\psi : A \to \Gamma$ is Frobenius if and only if so is $\psi \circ \phi$. On the other hand, it seems in general that one can not expect R to inherit any homological property from A. For instance, if R is an arbitrary finite dimensional algebra over a field k, then $\mathrm{End}_k(R)$ is a Frobenius algebra and the canonical ring homomorphism $R \to \mathrm{End}_k(R), x \mapsto (y \mapsto yx)$, is Frobenius.

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Centers of modular group algebras

Yoshihiro Otokita

This talk deals with the Loewy structure of the center of a modular group algebra.

Let G be a finite group and F an algebraically closed field of characteristic $p > 0$. We denote by $\mathcal{Z}$ the center of the group algebra FG. Then the dimension of $\mathcal{Z}$ is equal to the number of conjugacy classes of G. For a primitive idempotent b in $\mathcal{Z}$, $\mathcal{Z}b$ is the center of a block ideal of FG and it is a local algebra in the sense that the Jacobson radical $J(\mathcal{Z}b)$ has codimension 1. Our studies focus on the relation between $\mathcal{Z}b$ and *defect group* D. Then it is known that $D = \{1\}$ if and only if $J(\mathcal{Z}b) = 0$ (i.e. $\dim \mathcal{Z}b = 1$). In [1] Héthelyi and Külshammer conjectured that :

$$2\sqrt{p-1} \leq \dim \mathcal{Z}b \textit{ if } D \neq \{1\} \textit{ ?}$$

In order to solve this problem we examine the algebraic structure of $\mathcal{Z}b$. Our main theorem in this talk gives a lower bound for its Loewy length $LL(\mathcal{Z}b)$:

$$\frac{p^m + p - 2}{p - 1} \leq LL(\mathcal{Z}b) \textit{ where } p^m \textit{ is the exponent of the center of } D.$$

As a corollary, we conclude that $p + 2 \leq \dim \mathcal{Z}b$ provided $m \geq 2$.

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HOMOLOGICAL METHODS IN COMBINATORIAL COMMUTATIVE ALGEBRA

Kohji Yanagawa

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K.

Definition 1. For a simplicial complex $\Delta \subset 2^{\{1,\ldots,n\}}$, consider the monomial ideal $I_\Delta := (\prod_{i \in F} x_i \mid F \subset \{1, \ldots, n\}, F \not\in \Delta) \subset S$. We call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ.

Using this concept, R. Stanley proved Upper Bound Conjecture for simplicial spheres. Since then, this has been one of the central tools in the combinatorial study of finite simplicial complexes (c.f. [2]). Moreover, the algebraic study of Stanley-Reisner rings is still very active. It is a classical result that the Cohen–Macaulay-ness of $K[\Delta]$ only depends on the topology of the underlying space $|\Delta|$ (and $\mathrm{char}(K)$). For example, if $|\Delta|$ is homeomorphic to a sphere, then $K[\Delta]$ is Cohen–Macaulay for all K (moreover, it is Gorenstein).

In [3], I introduced the following notion, to use homological methods more systematically.

Definition 2. We say a finitely generated $\mathbb{Z}^n$-graded S-module M is *squarefree*, if it is $\mathbb{N}^n$ graded (i.e., $M_{\mathbf{a}} = 0$ for $\mathbf{a} \not\in \mathbb{N}^n$), and the multiplication map $M_{\mathbf{a}} \ni y \longmapsto x_i y \in M_{\mathbf{a}+\mathbf{e}_i}$ is bijective for all $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ and all i with $a_i > 0$.

Many modules appearing in the Stanley-Reisner ring theory are squarefree. At the same time, the class of squraefree modules is quite small. In fact, the category $\mathrm{Sq}\, S$ of squarefree S-modiles is equivalent to the category of finitely generated left modules over the incidence K-algebra of the poset $2^{\{1,\ldots,n\}}$.

The following are selected results on this notion.

- There is a duality functor $\mathbf{A} : \mathrm{Sq}\, S \to \mathrm{Sq}\, S$ such that $\mathbf{A}(K[\Delta]) \cong I_{\Delta^\vee}$, where $\Delta^\vee$ is the Alexander dual of Δ.
- $\mathbf{R}\,\mathrm{Hom}_S(-, \omega_S)$ gives a duality functor $\mathbf{D}$ from $D^b(\mathrm{Sq}\, S)$ to itself, where ω_S is the $\mathbb{Z}^n$-graded canonical module (it is just S itself as an underlying module). Moreover, we have $(\mathbf{A} \circ \mathbf{D})^3 \cong \mathbf{T}^{-2n}$, where $\mathbf{T}$ stands for the translation of $D^b(\mathrm{Sq}\, S)$.
- For $M \in \mathrm{Sq}\, S$, we have a constructible sheaf $M^\dagger$ on the $(n-1)$-simplex B. For example, $K[\Delta]^\dagger$ is the K-constant sheaf on $|\Delta| \subset B$ (more precisely, its direct image to B). In this context, the above mentioned duality $\mathbf{D}$ corresponds to Poincaré-Verdier duality on B.

On the other hand, I have begun to recognized that some "ring theoretic" arguments (e.g., Lefschetz property of artinian graded algebras, see [1]) are indispensable for further development of the Stanley-Reisner ring theory. Unfortunately, the compatibility between squarefree modules and such a ring theoretic argument is quite bad now. If time allows, I will discuss these problems.

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Finite dimensional algebras arising from dimer models and their derived equivalences

Yusuke Nakajima

A *dimer model* Γ is a bipartite graph described on the torus, and we can define the quiver Q as the dual of a dimer model. Precisely, we assign a vertex of Q dual to each face of Γ, an arrow of Q dual to each edge of Γ. For the quiver Q obtained as the dual of a "consistent" dimer model, the path algebra with certain relations, which is called the *Jacobian algebra*, has nice properties, for example it gives a *non-commutative crepant resolution* of a 3-dimensional Gorenstein toric singularity.

In the theory of dimer models, the notion of perfect matchings is so important. Here, we say that a subset D of edges of a dimer model Γ is a *perfect matching* if for any node n of Γ there is a unique edge in D containing n as the endpoint. We can also define a *perfect matching* of Q associated with Γ as the dual of D. By using a perfect matching D of Q, we define the degree d_D on each arrow $a \in Q_1$ of Q as

$$d_D(a) = \begin{cases} 1 & \text{if } a \in D \\ 0 & \text{otherwise} \end{cases}$$

and this makes the Jacobian algebra a graded algebra. Then, it is known that this graded Jacobian algebra arising from a consistent dimer model is a *bimodule* 3-*Calabi-Yau algebra of Gorenstein parameter* 1 [1, 2]. Moreover, if the degree zero part of such an algebra is finite dimensional, then it is a 2-*representation infinite algebra* (or *quasi* 2-*Fano algebra* in the context of noncommutative algebraic geometry) which is a generalization of a representation infinite hereditary algebra (see [3, 4]).

On the other hand, by using perfect matchings we can assign a lattice polygon Δ_Γ, which is called the *perfect matching polygon*, to each consistent dimer model Γ. In this talk, we first show that *internal perfect matchings* which correspond to interior lattice points of Δ_Γ give 2-representation infinite algebras.

Theorem 1 ([3]). *Let Q be the quiver obtained as the dual of a consistent dimer model Γ, and Λ be the graded Jacobian algebra whose degree is induced by a perfect matching D. Then, we see that D is an internal perfect matching of Q if and only if the degree zero part Λ_0 is a finite dimensional algebra, in which case Λ_0 is a 2-representation infinite algebra.*

After that, I also discuss the relationship between internal perfect matchings by using the *mutations of perfect matchings.*

Theorem 2 ([3]). *For a consistent dimer model Γ, internal perfect matchings of Γ are transformed into each other by the mutations if and only if they correspond to the same interior lattice point of Δ_Γ. When this is the case, 2-representation infinite algebras arising from these internal perfect matchings are derived equivalent.*

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Torsion classes for algebras with radical square zero

Toshitaka Aoki

Let A be a finite dimensional algebra over a field k. A torsion class of the category $\mathsf{mod}\,A$ of finitely generated right A-modules is a full subcategory which is closed under factor modules, extensions and isomorphisms. We denote by $\mathsf{tors}\,A$ the set of torsion classes of $\mathsf{mod}\,A$, then it forms a complete lattice with respect to inclusion [3]. In general, $\mathsf{tors}\,A$ can be decomposed into subsets $\mathsf{tors}_\epsilon\,A$ $(:= [\mathsf{Filt}\,(\mathrm{top}\,\epsilon A), \mathsf{Fac}\,\epsilon A])$, intervals associated with idempotents ϵ in A. We denote by $\mathsf{faithful\text{-}tors}\,A$ the set of *faithful torsion classes* of $\mathsf{mod}\,A$, that is, containing all injective modules. From the point of view of tilting theory, torsion classes have been extensively studied. Recently, it is shown in τ-tilting theory that a class of modules, called support τ-tilting modules, corresponds to functorially finite torsion classes. Note that tilting modules are precisely faithful support τ-tilting modules.

In this talk, we give a classification of torsion classes for an arbitrary algebra with radical square zero in terms of faithful torsion classes for hereditary algebras. A connection between the two classes of algebras were first studied by Gabriel [2]. Let A be a finite dimensional k-algebra with radical square zero. To a given idempotent ϵ in A we can attach a factor algebra A_ϵ of A, which is a hereditary algebra with radical square zero. Then we have the following result.

Theorem 1. *Let A be a finite dimensional k-algebra with radical square zero. For each idempotent ϵ in A, we have the following commutative diagram of partially ordered sets:*

$$\begin{tikzcd}
\mathsf{tors}_\epsilon\,A \arrow[r, "\sim"] & \mathsf{tors}_\epsilon\,A_\epsilon \arrow[r, "\sim"] & \mathsf{faithful\text{-}tors}\,A_\epsilon^{\mathrm{op}} \\
\mathrm{s}\tau\text{-}\mathsf{tilt}_\epsilon\,A \arrow[r, "\sim"] \arrow[u] & \mathrm{s}\tau\text{-}\mathsf{tilt}_\epsilon\,A_\epsilon \arrow[r, "\sim"] \arrow[u] & \mathsf{tilt}\,A_\epsilon^{\mathrm{op}} \arrow[u]
\end{tikzcd}$$

where $\mathrm{s}\tau\text{-}\mathsf{tilt}_\epsilon\,A$ (respectively, $\mathsf{tilt}\,A$) is the set of isomorphism classes of support τ-tilting modules corresponding to functorially finite torsion classes in $\mathsf{tors}_\epsilon\,A$ (respectively, tilting modules over A).

A key observation is that there is a reflection functor $\mathsf{mod}\,A_\epsilon \to \mathsf{mod}\,A_\epsilon^{\mathrm{op}}$. We can get numerical data on $\mathsf{tors}\,A$ through these isomorphisms. This result is a refinement of the previous work [1] from a categorical perspective.

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Auslander correspondence for triangulated categories

Norihiro Hanihara

The classical Auslander correspondence [2] 'for abelian categories' states a bijection between abelian categories with finitely many indecomposables and algebras with certain homological invariants, which provides an important viewpoint in representation theory. The aim of this talk is to give a triangulated analogue of this theorem. Namely, we give a homological characterization of triangulated categories satisfying some finiteness conditions. Let k be a field and $\mathcal{T}$ be a k-linear, Hom-finite, idempotent-complete triangulated category.

We consider two finiteness conditions. The first one is that $\mathcal{T}$ is *finite*, that is, $\mathcal{T}$ has an additive generator M. In this case, we call $\mathrm{End}_{\mathcal{T}}(M)$ the *Auslander algebra* of $\mathcal{T}$. We have the following homological characterization of Auslander algebras of finite triangulated categories:

Theorem 1. *Let k be a perfect field. Then, the following are equivalent for a basic finite dimensional algebra A.*

(1) *A is the Auslander algebra of a finite triangulated category.*
(2) *A is self-injective and there exists an automorphism α of A such that $\Omega^3 \simeq (-)_\alpha$ on $\underline{\mathrm{mod}}\, A$.*

Our proof depends on Amiot's result [1] as well as Green-Snashall-Solberg's [3], which plays an essential role in introducing a triangle strucure on certain additive categories.

The second finiteness condition is the following:

- There exists $M \in \mathcal{T}$ such that $\mathcal{T} = \mathrm{add}\{M[n] \mid n \in \mathbb{Z}\}$.
- For any $X, Y \in \mathcal{T}$, $\mathrm{Hom}_{\mathcal{T}}(X, Y[n]) = 0$ for almost all $n \in \mathbb{Z}$.

In this case, we say $\mathcal{T}$ is [1]-*finite*. For example, the bounded derived categories of representation-finite hereditary algerbas are [1]-finite. We have the following Auslander correspondence for [1]-finite triangulated categories:

Theorem 2. *Let k be an algebraically closed field. Then, there exists a bijection between the following.*

(1) *The set of triangle equivalence classes of* [1]*-finite algebraic triangulated categories.*
(2) *The set of graded Morita equivalence classes of finite dimensional graded self-injective algebras such that $\Omega^3 \simeq (-1)$.*
(3) *A disjoint union of Dynkin diagrams of type A, D, and E.*

A key step toward this bijection is, as is stated in (3), the classfication of [1]-finite algebraic triangulated categories, which is base on the tilting theorem due to Keller [5].

Applying this classification, we obtain the following result, which partially recovers an equivalene between the stable categories of simple singularities and the derived categories of Dynkin quivers.

Corollary 3. *Let k be an algebraically closed field and $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be a positively graded CM-finite Iwanaga-Gorenstein algebra such that each Λ_n is finite dimensional over k and Λ_0 has finite global dimension. Then, there exists a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\Lambda \simeq D^b(\mathrm{mod}\, kQ)$ for a disjoint union Q of Dynkin quivers.*

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THE CHAMBER STRUCTURES OF THE GROTHENDIECK GROUPS COMING FROM BRICKS

Sota Asai

Let A be a finite-dimensional algebra over a field K, and $\mathsf{mod}\,A$ be the category of finite-dimensional A-modules, and $\mathsf{proj}\,A \subset \mathsf{mod}\,A$ be the subcategory consisting of the projective A-modules.

For the Grothendieck groups $K_0(\mathsf{proj}\,A)$ and $K_0(\mathsf{mod}\,A)$, there exists a non-degenerate $\mathbb{Z}$-bilinear form $K_0(\mathsf{proj}\,A) \times K_0(\mathsf{mod}\,A) \to \mathbb{Z}$ called *Euler form*. The indecomposable projective A-modules and the simple A-modules give dual bases of $K_0(\mathsf{proj}\,A)$ and $K_0(\mathsf{mod}\,A)$ with respect to Euler form.

By [4, 6], there is a bijection from the set $\mathsf{2\text{-}silt}\,A$ of *2-term silting complexes* of the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$ to the set $\mathsf{2\text{-}smc}\,A$ of *2-term simple-minded collections* of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,A)$. Let n be the number of isomorphic classes of simple A-modules. If $T = \bigoplus_{i=1}^m T_i \in \mathsf{2\text{-}silt}\,A$ with all T_i indecomposable is sent to $\mathcal{X} = \{X_i\}_{i=1}^m \in \mathsf{2\text{-}smc}\,A$, then the *g-vectors* $[T_1], \dots, [T_n] \in K_0(\mathsf{proj}\,A)$ and the *c-vectors* $[X_1], \dots, [X_n] \in K_0(\mathsf{mod}\,A)$ form dual bases [1].

Now we consider the real-valued Grothendieck group $K_0(\mathsf{proj}\,A)_{\mathbb{R}} := K_0(\mathsf{proj}\,A) \otimes_{\mathbb{Z}} \mathbb{R}$. Then $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ is identified with the Euclidean space $\mathbb{R}^n$. We define the *cone* $C(T) \subset K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ for each $T \in \mathsf{2\text{-}silt}\,A$ by $C(T) := \{a_1[T_1] + \cdots + a_n[T_n] \mid a_1, \dots, a_n \in \mathbb{R}_{\geq 0}\}$. The cone $C(T)$ has n walls, and each wall corresponds to a mutation of T. The wall for the mutation of T at T_i is contained in the orthogonal space of $[X_i] \in K_0(\mathsf{mod}\,A)$ with respect to Euler form [2].

For $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, we can define the *numerical torsion class* $\mathcal{T}_\theta \subset \mathsf{mod}\,A$ as in [3]. By [7], if θ is an element in the interior of the cone $C(T)$ for $T \in \mathsf{2\text{-}silt}\,A$, then $\mathcal{T}_\theta$ coincides with the functorially finite torsion class $\mathcal{T}_T := \mathsf{Fac}\,H^0(T)$ corresponding to T in [1]. In particular, the set of θ satisfying $\mathcal{T}_\theta = \mathcal{T}_T$ coincides with $C(T)$ up to boundaries, and we can say that the numerical torsion class $\mathcal{T}_\theta$ changes in each time $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ leaps a wall of cones.

Moreover, $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ also gives the *semistable subcategory* $\mathcal{W}_\theta$ of $\mathsf{mod}\,A$ defined in [5]. It is an abelian subcategory of $\mathsf{mod}\,A$, so every simple object of $\mathcal{W}_\theta$ is a *brick*. Thus, for each brick S, we set $\Theta(S) := \{\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}} \mid S \in \mathcal{W}_\theta\}$ and consider the *chamber structure* of $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ with the walls given by $\Theta(S)$ for all bricks S. If A is τ-tilting finite, then each chamber is nothing but the cone for a 2-term silting complex. However, if A is not τ-tilting finite, then this observation is incomplete, because the Euclidean space $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$ is not covered by the cones for the 2-term silting complexes.

To solve this problem, we define an equivalence relation $\sim_{\mathcal{T}}$ called *$\mathcal{T}$-equivalence* by $\theta \sim_{\mathcal{T}} \theta' :\iff \mathcal{T}_\theta = \mathcal{T}_{\theta'}$. I have obtained the following two important results on the chamber structure of $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$.

Theorem 1. *The walls Θ_S determine the $\mathcal{T}$-equivalence classes.*

Theorem 2. *Let $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, then the interior of the $\mathcal{T}$-equivalence class of θ is not empty if and only if $\mathcal{T}_\theta$ is functorially finite.*

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Decorated Marked Surfaces

Qiu, Yu

Abstract: We introduce the decorated marked surface $\mathbf{S}_\triangle$ to study the associated Calabi-Yau-3 category $\mathcal{D}(\mathbf{S})$ of an unpunctured marked surface $\mathbf{S}$. In particular, we prove the following:

- [DMS I] There is a bijection X from the set $\mathrm{CA}(\mathbf{S}_\triangle)$ of closed arcs on $\mathbf{S}_\triangle$ to the set $\mathrm{Sph}\,\mathcal{D}(\mathbf{S})/[\mathbb{Z}]$ of shifts orbits of (reachable) spherical objects in $\mathcal{D}(\mathbf{S})$.
- [DMS I] X induces an isomorphism between the braid twist group $\mathrm{BT}(\mathbf{S}_\triangle)$, which is a subgroup of the mapping class group of $\mathbf{S}_\triangle$, to the spherical twist group $\mathrm{ST}\,\mathcal{D}(\mathbf{S})$.
- [DMS I(B)] There is a bijection from the set of open arcs on $\mathbf{S}_\triangle$ to a class of rigid indecomposable objects in the corresponding perfect category $\mathrm{per}\,\mathbf{S}$.
- [DMS II] The intersection number between arcs equals the dimension of $\mathrm{Hom}^\bullet$ between objects under the bijections above.
- [DMS III] The composition of Keller-Yang equivalences for a sequence of mutations in the surface is path-independent.
- [DMS IV] We give finite presentations of the group $\mathrm{BT}(\mathbf{S}_\triangle) \cong \mathrm{ST}\,\mathcal{D}(\mathbf{S})$ w.r.t. triangulations or quivers with potentials.
- [DMS V] The corresponding space $\mathrm{Stab}^\circ\,\mathcal{D}(\mathbf{S})$ of stability conditions is simply connected.

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Connectedness of the Balmer spectrum of the right bounded derived category of a commutative noetherian ring

Hiroki Matsui

This talk is based on the paper [2]. Tensor triangulated geonetry is a theory introduced by Balmer [3] to study tensor triangulated categories by algebro-geometric methods. Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category (i.e., a triangulated category equipped with a symmetric monoidal tensor product $\otimes$ which is compatible with the triangulated structure). Then we can define the notion of ideals, radical ideals, and prime ideals as in the case of commutative rings when we regard $\otimes$ as multiplication. For instance, a prime ideal of $\mathcal{T}$ is a proper thick subcategory $\mathcal{P}$ of $\mathcal{T}$ such that

(1) (ideal) $\forall X \in \mathcal{T}$, $\forall Y \in \mathcal{P}$, one has $X \otimes Y \in \mathcal{P}$, and
(2) (prime) if $X \otimes Y \in \mathcal{T}$, then $X \in \mathcal{P}$ or $X \in \mathcal{P}$.

Then Balmer defines a topological space $\mathsf{Spc}\mathcal{T}$ as the set of all prime ideals of $\mathcal{T}$ with Zariski topology. Balmer's celebrated result says that

Theorem 1 (Balmer)**.** *There is an order-preserving one-to-one correspondence between*

(1) *the set of radical thick tensor ideals of* $\mathcal{T}$ *and*
(2) *the set of Thomason subsets of* $\mathsf{Spc}\mathcal{T}$*.*

From this result, if we want to classify the radical thick tensor ideals of $\mathcal{T}$, we have only to understand the topological structure of $\mathsf{Spc}\mathcal{T}$.

In this talk, we consider the right bounded derived category $\mathsf{D}^-(\mathsf{mod}R)$ of a commutative noetherian ring R. This triangulated category is a tensor triangulated category with respect to derived tensor product $\otimes^{\mathsf{L}}_R$, and we can consider its Balmer spectrum $\mathsf{Spc}\mathsf{D}^-(\mathsf{mod}R)$. We discuss some topological structures of this topological space $\mathsf{Spc}\mathsf{D}^-(\mathsf{mod}R)$. The following theorem is one of the main result in this talk.

Theorem 2. *The Balmer spectrum* $\mathsf{Spc}\mathsf{D}^-(\mathsf{mod}R)$ *is connected (irreducible) if and only if so is the Zariski spectrum* $\mathsf{Spec}R$*.*

The key to prove our main theorem is the classification of thick tensor ideals of $\mathsf{D}^-(\mathsf{mod}R)$ generated by bounded complexes given in [3].

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Dimensions of singular categories of hypersurfaces of countable representation type

Tokuji Araya, Kei-ichiro Iima, Maiko Ono, Ryo Takahashi

Throughout this talk, let k be an algebraically closed field of characteristic zero, and let R be a complete local hypersurface over k with countable representation type. Denote by $\mathrm{D_{sg}}(R)$ the singularity category of R, and let $\mathrm{D_{sg}^o}(R)$ be the full subcategory of $\mathrm{D_{sg}}(R)$ consisting of objects locally zero on the punctured spectrum.

Let $\mathcal{T}$ be a triangulated category. For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ which is closed under isomorphisms, shifts, finite sums and summands. For subcategories $\mathcal{X}, \mathcal{Y}$ of $\mathcal{T}$ we denote by $\mathcal{X} * \mathcal{Y}$ the subcategory consisting of objects $M \in \mathcal{T}$ such that there is an exact triangle $X \to M \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Set $\mathcal{X} \diamond \mathcal{Y} := \langle \langle \mathcal{X} \rangle * \langle \mathcal{Y} \rangle \rangle$. For a subcategory $\mathcal{X}$ of $\mathcal{T}$ we put $\langle \mathcal{X} \rangle_0 := 0$, $\langle \mathcal{X} \rangle_1 := \langle \mathcal{X} \rangle$, and inductively define $\langle \mathcal{X} \rangle_n := \mathcal{X} \diamond \langle \mathcal{X} \rangle_{n-1}$ for $n \geq 2$. The *dimension* of $\mathcal{T}$ *with respect to* $\mathcal{X}$, denote by $\dim_{\mathcal{X}} \mathcal{T}$, is the infimum of $\{n \geq 0 \mid \mathcal{T} = \langle \mathcal{X} \rangle_{n+1}\}$.

The main results in this talk are the following two theorems.

Theorem 1. *For all nonzero objects M of $\mathrm{D_{sg}}(R)$, the residue field k belongs to $\langle \{M\} \rangle_2^{\mathrm{D_{sg}}(R)}$.*

Theorem 2. *Let $\mathcal{T}$ be a nonzero thick subcategory of $\mathrm{D_{sg}}(R)$, and let $\mathcal{X}$ be a full subcategory of $\mathcal{T}$. Then the following statements hold.*

(1) *$\mathcal{T}$ coincides with either $\mathrm{D_{sg}}(R)$ or $\mathrm{D_{sg}^o}(R)$.*

(2) (a) *If $\mathcal{T} = \mathrm{D_{sg}}(R)$, then*

$$\dim_{\mathcal{X}} \mathcal{T} = \begin{cases} 0 & (\langle \mathcal{X} \rangle = \mathcal{T}), \\ 1 & (\langle \mathcal{X} \rangle \neq \mathcal{T},\ \langle \mathcal{X} \rangle \nsubseteq \mathrm{D_{sg}^o}(R)), \\ \infty & (\langle \mathcal{X} \rangle \subseteq \mathrm{D_{sg}^o}(R)). \end{cases}$$

(b) *If $\mathcal{T} = \mathrm{D_{sg}^o}(R)$, then*

$$\dim_{\mathcal{X}} \mathcal{T} = \begin{cases} 0 & (\langle \mathcal{X} \rangle = \mathcal{T}), \\ 1 & (\langle \mathcal{X} \rangle \neq \mathcal{T},\ \#\mathrm{ind}\langle \mathcal{X} \rangle = \infty), \\ \infty & (\#\mathrm{ind}\langle \mathcal{X} \rangle < \infty). \end{cases}$$

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Resolution of DG-modules and their applications for commutative DG-algebras

Hiroyuki Minamoto

Differential graded (DG) algebra lies in the center of homological algebra and allows us to use techniques of homological algebra of ordinary algebras in much wider context. Projective resolutions and injective resolutions which are the fundamental tools of homological algebra already have their DG-versions, which are called a DG-projective resolution and a DG-injective resolution. The aim of this talk is to introduce a different DG-versions of projective resolution (sup-projective resolution) and injective resolutions (inf-injective resolutions) for DG-modules over a connective DG-algebra.

Recall that a (cohomological) DG-algebra R is called *connective* if the vanishing condition $\mathrm{H}^{>0}(R) = 0$ of the cohomology groups is satisfied. There are rich sources of connective DG-algebras: the Koszul algebra $K_R(x_1, \cdots, x_d)$ in commutative ring theory, and an endomorphism DG-algebra $\mathbb{R}\mathrm{Hom}(S, S)$ of a silting object S. We would like to point out that a commutative connective DG-algebras are regarded as the coordinate algebras of derived affine schemes in derived algebraic geometry (see e.g. [1]).

The motivation for this work came from the projective dimensions and the injective dimensions for DG-modules introduced by Yekutieli. In the paper [3] he introduced projective dimension and injective dimension of DG-modules by generalizing the characterization of projective dimension and injective dimension of ordinary modules by vanishing of Ext-group. An important feature of newly introduced resolutions is that, roughly speaking, the "length" of these resolutions give projective or injective dimensions.

Theorem 1. *Let R be a connective DG-algebra. Then a DG-R-module M is of $\mathrm{pd}_R M = d$ if and only if it has a sup-projective resolution $P_\bullet$ of length e such that*

$$d = e + \sup M - \sup P_e.$$

Other conditions for $\mathrm{pd}M = d$ will be given in the talk or the poster.

We show that these resolutions allows us to investigate basic properties of projective and injective dimensions of DG-modules. As an application we introduce the global dimension of a connective DG-algebra and show that finiteness of global dimension is derived invariant.

If time permits, we will discuss a commutative DG-algebra. We observe that a DG-counter part $E_R(R/\mathfrak{p})$ of the class of indecomposable injective modules are parametrized by prime ideals $\mathfrak{p} \in \mathrm{Spec}\mathrm{H}^0(R)$ of the 0-th cohomology algebra. This fact is compatible with the view point of derived algebraic geometry that the base affine scheme of the derived affine scheme $\mathrm{Spec}R$ associated to a CDGA R is the affine scheme $\mathrm{Spec}\mathrm{H}^0(R)$. We give a structure theorem of minimal injective resolution of dualizng complex.

This talk is a report of my paper [2].

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Relations for Grothendieck groups and representation-finiteness

Haruhisa Enomoto

A *Grothendieck group* is the abelian group associated with an exact category. It was known that the defining relations of the Grothendieck group is closely related to the representation-finiteness. My aim of this talk is to unify and generalize several known results about this.

Let $\mathcal{E}$ be a Krull-Schmidt exact category. A Grothendieck group $\mathsf{K}_0(\mathcal{E})$ of $\mathcal{E}$ is defined to be the quotient group $\mathsf{K}_0(\mathcal{E}, 0)/\mathsf{Ex}(\mathcal{E})$, where $\mathsf{K}_0(\mathcal{E}, 0)$ is the free abelian group with basis $\mathsf{ind}\mathcal{E}$ (the set of isomorphism classes of indecomposables in $\mathcal{E}$) and $\mathsf{Ex}(\mathcal{E})$ is the subgroup of $\mathsf{K}_0(\mathcal{E})$ generated by

$$\{[X] - [Y] + [Z] \mid \text{there exists a short exact sequence } 0 \to X \to Y \to Z \to 0 \text{ in } \mathcal{E}\}.$$

Among short exact sequences, *AR sequences* are minimal in some sense, and have played an essential role in the representation theory of algebras and commutative rings. We denote by $\mathsf{AR}(\mathcal{E})$ the subgroup of $\mathsf{Ex}(\mathcal{E})$ generated by AR sequences.

For an artin algebra Λ, it was proved by Butler [Bu] and Auslander [Au] that $\mathsf{AR}(\mathsf{mod}\,\Lambda) = \mathsf{Ex}(\mathsf{mod}\,\Lambda)$ holds if and only if $\mathsf{mod}\,\Lambda$ has finitely many indecomposables. Similar results were obtained for certain subcategories of $\mathsf{mod}\,\Lambda$, e.g. [MMP, PR]. Our result about artin algebras is the following, which generalizes all of these results.

Theorem 1. *Let Λ be an artin algebra and $\mathcal{E}$ a contravariantly finite resolving subcategory of $\mathsf{mod}\,\Lambda$. Then $\mathcal{E}$ has finitely many indecomposables if and only if $\mathsf{AR}(\mathcal{E}) = \mathsf{Ex}(\mathcal{E})$ holds.*

Next I will discuss what happens if we drop the assumption of artin-ness. Similar equivalences was proved for the category $\mathsf{CM}\,\Lambda$ of Λ-lattices over an order Λ under some restrictions: [AR, Hi], and we have a partial result [Ko]. I will give a partial result on this which generalizes these results:

Theorem 2. *Let R be a complete Cohen-Macaulay local ring and Λ an R-order. Then the following holds.*

(1) *If $\mathsf{CM}\,\Lambda$ has finitely many indecomposables, then $\mathsf{AR}(\mathsf{CM}\,\Lambda) = \mathsf{Ex}(\mathsf{CM}\,\Lambda)$ holds.*

Suppose that $\mathsf{AR}(\mathsf{CM}\,\Lambda) = \mathsf{Ex}(\mathsf{CM}\,\Lambda)$ holds. Then we have the following.

(2) *$\Omega\mathsf{CM}\,\Lambda$, the category of syzygies of modules in $\mathsf{CM}\,\Lambda$, has finitely many indecomposables.*

(3) *Assume that Λ is a Gorenstein order or has finite global dimension. Then $\mathsf{CM}\,\Lambda$ has finitely many indecomposables.*

We use the functorial method to prove these results. If time permits, I will give a general relations, which is used to prove above, between the conditions: (a) having finitely many indecomposables, (b) some functorial condition and (c) AR=Ex.

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Sortable elements and torsion pairs for quivers

Yuya Mizuno

Path algebras are one of the most fundamental and important classes of algebras. In this talk, we discuss torsion pairs for the module category of a path algebra, and explain a close relationship between torsion pairs and the elements of the Coxeter group of the quiver.

Let Q be acyclic quiver and W the Coxeter group of Q. Then the result of [ORT] asserts that there exists a bijection between the elements of W and the set of cofinite quotient-closed subcategories of $\mathrm{mod}kQ$. Thus, for a given $w \in W$, we can give a cofinite quotient-closed category $\mathcal{C}_w$. Then we pose the following natural questions.

Question: (1) When is $\mathcal{C}_w$ a torsion class of $\mathrm{mod}kQ$ for $w \in W$?
(2) When $\mathcal{C}_w$ is a torsion class, how can we describe the corresponding torsion free class ?

In this talk, we will give an answer for the above question. Our method is the theory of preprojective algebra Π of Q. The result [BIRS] allows a connection between the representation theory of Π and W. Using this connection, we parametrize torsion pairs of $\mathrm{mod}kQ$ by some elements of W, called sortable elements, and we explain a conjecture by [ORT].

This is based on our work [MT].

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A geometric model of Brauer graph algebras

Takahide Adachi and Aaron Chan

Given a surface, we can construct various triangulated categories which are usually motivated by the homological mirror symmetry conjecture and related Calabi-Yau algebras/categories. In the constructions, curves on a surface frequently give rise to an important class of objects. Since a Brauer graph can be embedded into a certain surface with marked points and boundary, we can also ask if curves of the surface give us any interesting objects in the triangulated categories associated to a Brauer graph algebra. Generalizing ideas from works by Khovanov-Seidel [1], Seidel-Thomas [3] and Marsh-Schroll [2], we associate curves on the surface with complexes in the bounded homotopy category. In this talk, we give details on this construction and show how one can interpret some homological phenomenons and problems of a Brauer graph algebra using the combinatorics of the associated surface.

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Cyclotomic polynomials

Kaoru Motose

In this talk, I present about relationship between the fundamental results about cyclotomic polynomials and the next mathematical items, 1. Order, 2. Decomposition of cyclotomic polynomials over fields, 3. Feit Thomphson conjecture, and 4. Rational primes.

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Batalin-Vilkovisky algebra structures on the Hochschild cohomology of self-injective Nakayama algebras

Tomohiro Itagaki

In this talk, we give Batalin-Vilkovisky (BV) algebra structures on the Hochschild cohomology of self-injective Nakayama algebras over an algebraically closed field.

Tradler [2] discovered that Hochschild cohomology of arbitrary symmetric algebra has a BV algebra structure given by a symmetric bilinear form. Later, Lambre, Zhou and Zimmermann [1] discovered that Hochschild cohomology of Frobenius algebras with diagonalizable Nakayama automorphism has a BV algebra structure. However, it is not known that Hochschild cohomology of Frobenius algebras has a BV algebra structure in general, and there are few examples of complete calculation of BV differentials on Hochschild cohomology of Frobenius algebras which are not symmetric.

Recently, for any Frobenius algebra A, Volkov [3] defined the cohomology $\mathrm{HH}^*(A)^{\nu\uparrow}$ of Hochschild complex related to Nakayama automorphism ν, which induces Gerstenhaber algebra $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$. Moreover, Volkov also found a BV algebra structure on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$. In particular, if the Nakayama automorphism ν is diagonalizable, then $\mathrm{HH}^*(A)^{\nu\uparrow} \cong \mathrm{HH}^*(A)$ and the BV differential on $(\mathrm{HH}^*(A)^{\nu\uparrow}, \smile, [\ ,\ ])$ induces the one on the Gerstenhaber algebra $(\mathrm{HH}^*(A), \smile, [\ ,\ ])$.

We give BV differentials on the Hochschild cohomology of self-injective Nakayama algebras by dividing the computation into two cases: (a) the characteristic of the ground field does not divide the order of the Nakayama automorphism; (b) the characteristic of the ground field divides the order of the Nakayama automorphism. For a self-injective Nakayama algebra Λ in case (b), by computing $\mathrm{HH}^*(\Lambda)^{\nu\uparrow}$ and BV differentials on $(\mathrm{HH}^*(\Lambda)^{\nu\uparrow}, \smile, [\ ,\ ])$, we have $\mathrm{HH}^*(\Lambda)^{\nu\uparrow} \cong \mathrm{HH}^*(\Lambda)$ and $[\ ,\ ] = 0$. On the other hand, $[\ ,\ ] \neq 0$ on Hochschild cohomology of Λ in case (a) in general.

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An application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring

Kazunori Nakamoto and Takeshi Torii

Let us begin with the definition of the moduli of subalgebras of the full matrix ring.

Definition 1. We say that a subsheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ is a *mold* of degree n on a scheme X if $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ is a locally free sheaf. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf.

Proposition 2. *The following cotravariant functor is representable by a closed subscheme of the Grassmann scheme* $\mathrm{Grass}(d, n^2)$*:*

$$\begin{array}{cccccc} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \;\middle|\; \mathcal{A} \textit{ is a rank } d \textit{ mold of degree } n \textit{ on } X \right\}. \end{array}$$

Let $\mathcal{A}$ be the universal mold on $\mathrm{Mold}_{n,d}$. For $x \in \mathrm{Mold}_{n,d}$, set $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_{\mathrm{Mold}_{n,d}}} k(x)$, where $k(x)$ is the residue field of x. We describe the dimension of the Zariski tangent space $T_x\mathrm{Mold}_{n,d}$ and the smoothness of $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ at x by using Hochschild cohomology $H^i(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x))$.

Theorem 3. *Set* $N(\mathcal{A}(x)) := \{Y \in \mathrm{M}_n(k(x)) \mid [X, Y] \in \mathcal{A}(x) \textit{ for } X \in \mathcal{A}(x)\}$. *The dimension of the Zariski tangent space* $T_x\mathrm{Mold}_{n,d}$ *is given by*

$$\dim T_x\mathrm{Mold}_{n,d} = \dim H^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) + n^2 - \dim N(\mathcal{A}(x)).$$

By using cohomology classes of $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x))$, we can describe the smoothness of $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ at x. In particular, we have

Theorem 4. *If* $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$*, then* $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ *is smooth at* x.

Let A be an R-subalgebra of $\mathrm{M}_n(R)$ over a commutative ring R. Assume that $\mathrm{M}_n(R)/A$ is a projective R-module. We introduce several results on $H^i(A, \mathrm{M}_n(R)/A)$.

Let $Q = (Q_0, Q_1)$ be an ordered quiver. For $a, b \in Q_0$, we say that $a \geq b$ if $a = b$ or there exists an oriented path from b to a. The incidence algebra RQ/I can be written by $\Lambda = \oplus_{a \geq b} Re_{ba}$. Fix a numbering $Q_0 = \{1, 2, \ldots, n\}$. By regarding e_{ba} as $E_{ba} \in \mathrm{M}_n(R)$, Λ can be identified with the R-subalgebra $\oplus_{a \geq b} RE_{ba}$ of $\mathrm{M}_n(R)$. In a similar way as [1], we have the following theorem:

Theorem 5 ([2])**.** *Let* Λ *be as above. Then* $H^i(\Lambda, \mathrm{M}_n(R)/\Lambda) = 0$ *for* $i \geq 0$.

Theorem 6 ([2])**.** *For* $n_1 + \cdots + n_r = n$, *we set* $\mathcal{P}_{n_1, n_2, \ldots, n_r}(R) = \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \textit{ if } \sum_{k=1}^{s} n_k < i \leq \sum_{k=1}^{s+1} n_k \textit{ and } j \leq \sum_{k=1}^{s} n_k\}$. *Then* $H^i(\mathcal{P}_{n_1, n_2, \ldots, n_r}(R), \mathrm{M}_n(R)/\mathcal{P}_{n_1, n_2, \ldots, n_r}(R)) = 0$ *for* $i \geq 0$.

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Some remarks on Avella-Alaminos-Geiss invariants of gentle algebras

Hiroyuki Nakaoka

In [1], Avella-Alaminos and Geiss have introduced derived invariants $\phi_A \colon \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ for gentle algebras $A = KQ/I$. Since these invariants can be calculated combinatorially from the bound quiver (Q, I), they are effectively used in the classifiation of gentle algebras with some conditions, e.g. in [2]. They also have relation with the dimensions of the Hochschild cohomologies $HH^n(A)$, as shown in [3] and [4].

In this talk, I would like to introduce its definition and properties from related literatures, and possibly give an experimental construction of algebras related to these invariants.

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On projective modules with unique maximal submodule
唯一の極大部分加群を持つ射影加群について

Masahisa Sato
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INTRODUCTION

It passed 47 years after R.Ware gave the following problem in *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** (1971), 233-256.;

Let R be a ring and P a projective right R-module with unique maximal submodule L, then L is the largest submodule of P.

In this talk, we report affirmative answer for this problem.

Before we solve this problem, we show the following fact;

Any projective module has a maximal submodule.

This implies the following fact which generalizes Nakaya-Azumaya Lemma for projective modules.

If $PJ(R) = P$ for a projective module P, then $P = 0$.

ON PRIMITIVE RINGS AND IDEALS

We review the definition of primitive rings and primitive ideals and their basic properties.

Definition 1. A ring R is called a right primitive ring if there is a faithful simple right R-module. A two sided ideal T of R is called a primitive right ideal if the factor algebra R/T is a primitive ring.

The Jacobson radical $J(R)$ of R is the intersection of all maximal right ideals of R. This is same as the intersection of all primitive right ideals of R.

Let M be a right R-module and S a subset of M. An (right) annihilator of S is defined by $\mathrm{Ann}_R(S) = \{r \in R \mid Sr = 0\}$.

Remark 2. A primitive right ideal T with a faithful simple right R/T-module R/J is given by the form $T = \mathrm{Ann}_R(R/J)$. Hence T is maximal between two sided ideals included in J.

Also $T = \bigcap_{I \in \Gamma} I = \bigcap_{I \in \Delta} I$, here Γ is the set consisting of a maximal right ideal I with $T \subset I$ and Δ is the set consisting of a maximal right ideal I with $R/J \cong R/I$.

STRUCTURE THEOREM

The following is a key theorem to solve R.Ware's problem.

Proposition 3. *Let R be a ring and P a projective right R-module with unique maximal submodule L, then P is indecomposable or there are direct summands P_1 and P_2 such that $P = P_1 \oplus P_2$, P_1 has unique maximal submodule and P_2 does not have any maximal submodules.*

EXAMPLES

We give the following examples.

(1) Non-projective indecomposable module with unique maximal submodule but not largest submodule. i.e., Ware's problem is not ture for non-projective modules in general.

(2) A non-projective module M with the properties $MJ \neq M$ for only one maximal right ideal J and $MI = M$ for any maximal right ideal $I \neq J$.

(3) An infinitely generated projective module P with the properties $PJ \neq P$ for only one maximal right ideal J and $PI = P$ for any maximal right ideal $I \neq J$.

On (finite) Σ-Rickart modules:
On a module theoretic setting of the (semi-)hereditary property of rings

Gangyong Lee* and Mauricio Medina-Bárcenas
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After Kaplansky introduced hereditary rings in the earliest 50's, they have been extensively investigated in the literature. Hereditary rings have been characterized in different ways, the most common of them is that given in [1, Ch.I, Theorem 5.4]: a ring R is right hereditary if and only if every submodule of any projective right R-module is projective if and only if every factor module of any injective right R-module is injective. Semi-hereditary rings also ware characterized as a similar way as hereditary rings.

In this talk, we introduce the notion of (finite) Σ-Rickart modules by utilizing the endomorphism ring of a module and by using the recent notion of Rickart modules [2] as a module theoretic analogue of a right (semi-)hereditary ring. A module M is called *(finite) Σ-Rickart* if every (finite) direct sum of copies of M is Rickart ([3, Definition 2.21]). It is shown that any direct summand and any (finite) direct sum of copies of a (finite) Σ-Rickart module are Σ-Rickart modules. Also, we provide several characterizations of (finite) Σ-Rickart modules which are including generalizations of the most common results (see the above results for rings) of (semi-)hereditary rings in a module theoretic setting. Also, we have a characterization of a finitely generated (finite) Σ-Rickart module in terms of its endomorphism ring.

This talk is based on a joint work with Mauricio-Bárcenas.

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A strongly quasi-hereditary structure on Auslander–Dlab–Ringel algebras

Mayu Tsukamoto

Ringel introduced special classes of quasi-hereditary algebras called left-strongly quasi-hereditary algebras and strongly quasi-hereditary algebras [5]. It is known that left-strongly quasi-hereditary algebras have better upper bound for global dimension than that of arbitrary quasi-hereditary algebras.

Left-strongly (resp. strongly) quasi-hereditary algebras are strongly related to rejective chains defined below. Let $\mathcal{C}$ be a Krull–Schmidt category. A chain $\mathcal{C} = \mathcal{C}_0 \supset \mathcal{C}_1 \supset \cdots \supset \mathcal{C}_n = 0$ of subcategories of $\mathcal{C}$ is called a *total left rejective chain* if the following conditions hold for $1 \leq i \leq n$:

(a) $\mathcal{C}_i$ is a *left rejective subcategory* of $\mathcal{C}$ (*i.e.*, for any $X \in \mathcal{C}$, there exists an epic left $\mathcal{C}_i$-approximation $f^X \in \mathcal{C}(X, Y)$ of X);

(b) the Jacobson radical of the factor category $\mathcal{C}_{i-1}/[\mathcal{C}_i]$ is zero.

Dually, a total right rejective chain is defined. A rejective chain is defined as a total left rejective chain and a total right rejective chain.

Proposition 1 ([6, Theorem 3.22]). *Let A be an artin algebra. Then A is left-strongly* (*resp. strongly*) *quasi-hereditary if and only if the category* $\mathsf{proj}A$ *has a total left rejective* (*resp. rejective*) *chain.*

Let A be an artin algebra with Loewy length m and J the Jacobson radical of A. Auslander proved that the endomorphism algebra $B := \mathrm{End}_A(\bigoplus_{i=1}^m A/J^i)$ has finite global dimension [1]. Moreover, Dlab and Ringel showed that B is quasi-hereditary [3]. Hence B is called an Auslander–Dlab–Ringel (ADR) algebra. Recently, Conde gave a left-strongly quasi-hereditary algebra structure on ADR algebras [2].

In this talk, we study ADR algebras of semilocal modules introduced by Lin and Xi [4]. Recall that a module M is said to be *semilocal* if M is a direct sum of modules which have a simple top. Since any artin algebra is a semilocal module, the ADR algebras of semilocal modules are a generalization of the original ADR algebras. In [4], they proved that ADR algebras of semilocal modules are quasi-hereditary. We refine this result by using Proposition 1.

Theorem 2 ([7, Theorem 2.2]). *Let A be an artin algebra, M a semilocal A-module with Loewy length m and J the Jacobson radical of A. Then the category $\mathsf{add}(\bigoplus_{i=1}^m M/MJ^i)$ has a total left rejective chain. In particular, the ADR algebra $\mathrm{End}_A(\bigoplus_{i=1}^m M/MJ^i)$ is left-strongly quasi-hereditary.*

As an application, we give a tightly upper bound for global dimension of an ADR algebra.

It is known that the global dimension of any strongly quasi-hereditary algebra is at most two [5, Proposition A.2]. We note that algebras with global dimension at most two are not always strongly quasi-hereditary. We prove that the converse holds if B is an original ADR algebra.

Theorem 3 ([7, Theorem 3.1]). *Let A be an artin algebra with Loewy length $m \geq 2$ and J the Jacobson radical of A. Let $B := \mathrm{End}_A(\bigoplus_{i=1}^m A/J^i)$ be the ADR algebra of A. Then B is strongly quasi-hereditary if and only if* $\mathrm{gl.dim}B = 2$ *holds.*

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Components of the stable Auslander-Reiten quiver for a symmetric order over a complete discrete valuation ring

Kengo Miyamoto

In representation theory of algebras, we often use Auslander–Reiten theory to analysis various additive categories arising in representation theory because we may prove many important combinatorial and homological properties with the help of the theory. For important classes of finite dimensional algebras, there exist strong restrictions on stable Auslander–Reiten quivers, for example see [2, 6, 7, 8]. However, if the base ring is not a field but a complete regular local ring, then the shape of (stable) Auslander–Reiten components for algebras are mostly unknown.

Let $\mathcal{O}$ be a complete discrete valuation ring, κ is the residue field, $\mathcal{K}$ the quotient field and A an $\mathcal{O}$-order. We denote by $\mathsf{latt}\text{-}A$ the category consisting of A-lattices. Then, the category $\mathsf{latt}\text{-}A$ admits almost split sequences if and only if A is an isolated singularity. In this case, one can find some results on the shape of Auslander–Reiten components, for example [3]. Otherwise, we have to consider a suitable full subcategory of $\mathsf{latt}\text{-}A$ which admits almost split sequences. In [1], we considered the full subcategory of $\mathsf{latt}\text{-}A$, say $\mathsf{latt}^{(\natural)}\text{-}A$, consisting of A-lattices M such that $M \otimes_{\mathcal{O}} \mathcal{K}$ is projective as an $A \otimes_{\mathcal{O}} \mathcal{K}$-module, and we defined the concept of the stable Auslander–Reiten quiver for $\mathsf{latt}^{(\natural)}\text{-}A$, for a symmetric $\mathcal{O}$-order A. The following is the main results.

Theorem 1 ([1, 4])**.** *Let A be a symmetric $\mathcal{O}$-order and $\mathcal{C}$ be a stable AR component for $\mathsf{latt}^{(\natural)}\text{-}A$ with infinitely many vertices. Then, the following statements hold.*

(1) *Suppose that $\mathcal{C}$ is periodic. Then, one of the following statements holds:*
 (i) *If $\mathcal{C}$ has no loops, then the tree class is one of infinite Dynkin diagrams.*
 (ii) *If $\mathcal{C}$ has loops, then $\mathcal{C} \setminus \{loops\} = \mathbb{Z}A_\infty/\langle\tau\rangle$, and the loops appear on the boundary of $\mathcal{C}$.*

(2) *Suppose that $\mathcal{C}$ is non-periodic. Then, $\mathcal{C}$ has no loops. Moreover, if either $\mathcal{C}$ does not contain Heller lattices or $A \otimes_{\mathcal{O}} \kappa$ has finite representation type, then the tree class of $\mathcal{C}$ is one of infinite Dynkin diagrams or Euclidean diagrams.*

By applying this restriction to the symmetric Kronecker algebra A, we have the following.

Theorem 2 ([4, 5])**.** *Let A be the symmetric Kronecker algebra over $\mathcal{O}$, and let $\mathcal{C}$ be a stable AR component for $\mathsf{latt}^{(\natural)}\text{-}A$ that contains a Heller lattice. If $\mathcal{C}$ is non-periodic, then $\mathcal{C} \simeq \mathbb{Z}A_\infty$. Otherwise, $\mathcal{C}$ is isomorphic to $\mathbb{Z}A_\infty/\langle\tau\rangle$ if the Heller lattice is given from strings, else $\mathbb{Z}A_\infty/\langle\tau^2\rangle$.*

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$\mathbb{X}$-Stability Conditions on Calabi-Yau-$\mathbb{X}$ Categories

Qiu, Yu

Abstract: We introduce the notion of $\mathbb{X}$-stability condition, consisting of a Bridgeland stability condition and a complex number s, on a Calabi-Yau-$\mathbb{X}$ category $\mathcal{D}_{\mathbb{X}}$ satisfying the equation

$$\mathbb{X}(\sigma) = s \cdot \sigma. \tag{1}$$

Here, the Grothendieck group of $\mathcal{D}_{\mathbb{X}}$ is

$$K(\mathcal{D}_{\mathbb{X}}) \cong R^{\oplus n}, \quad R = \mathbb{Z}[q^{\pm 1}]$$

and the R-structure on $K(\mathcal{D}_{\mathbb{X}})$ is provided by the auto-equivalence $\mathbb{X}$. This gives the q-deformation of Bridgeland's stability condition, in the sense that when fixing s, the space $\mathbb{X}\mathrm{Stab}_s \mathcal{D}_{\mathbb{X}}$ of $\mathbb{X}$-stability conditions (σ, s) is a complex manifold with dimension n.

We also introduce the $\mathbb{R}$-generalization of global dimension for algebras, the global dimension function on a stability condition $\sigma = (Z, \mathcal{P})$:

$$\operatorname{gldim} \sigma = \sup\{\phi_2 - \phi_1 \mid \operatorname{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0\}, \tag{2}$$

where $\mathcal{P}$ is the slicing (a $\mathbb{R}$-collection of t-structures). We give a criterion

$$\operatorname{gldim} \widehat{\sigma} + 1 \leq \operatorname{Re}(s)$$

of constructing an $\mathbb{X}$-stability condition on $\mathcal{D}_{\mathbb{X}}$ from an usual stability condition $\widehat{\sigma}$ on the $\mathbb{X}$-heart $\mathcal{D}_\infty$ of $\mathcal{D}_{\mathbb{X}}$.

We discuss motivation/application that relates to Saito-Frobenius structure, mirror symmetry and cluster theory.

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On certain Morita invariants involving commutator subspace and radical powers

Shigeo Koshitani and Taro Sakurai

In 1941, Brauer-Nesbitt [2] established a characterization of a block of a finite group with trivial defect group as a block B with $k(B) = 1$ where $k(B)$ is the number of irreducible ordinary characters of B. In 1982, Brandt [1] established a characterization of a block of a finite group with defect group of order two as a block B with $k(B) = 2$. These correspond to the cases when the block is Morita equivalent to the one-dimensional algebra and to the non-semisimple two-dimensional algebra, respectively [5].

In this talk, we redefine $k(A)$ to be the codimension of the commutator subspace $K(A)$ of a finite-dimensional algebra A and prove analogous statements for arbitrary (not necessarily symmetric) finite-dimensional algebras.

Theorem 1 (Chlebowitz [3], Koshitani-Sakurai [4]). *Suppose F is a splitting field for A. Then the following holds.*

(i) $k(A) = 1 \iff \mathsf{mod}\, A \simeq \mathsf{mod}\, F$.

(ii) $k(A) = 2$ *and* $\ell(A) = 1 \iff \mathsf{mod}\, A \simeq \mathsf{mod}\, F[X]/(X^2)$.

This is achieved by extending the Okuyama refinement [6] of the Brandt result to this setting (part (ii) of Theorem 2; see also Shimizu [8]).

To this end, we study the codimension of the sum of the commutator subspace $K(A)$ and nth Jacobson radical $\mathrm{Rad}^n(A)$, which is denoted by $KR^n(A) = K(A) + \mathrm{Rad}^n(A)$.

Theorem 2 (Koshitani-Sakurai [4]). *Let $\{\, S_i \mid 1 \le i \le \ell(A) \,\}$ be a complete set of pairwise non-isomorphic simple right A-modules and let C_A be the Cartan matrix of A. Suppose F is a splitting field for A. Then the following holds.*

(i) $\operatorname{codim} KR^1(A) = \ell(A)$.

(ii) $\operatorname{codim} KR^2(A) = \ell(A) + \sum_{i=1}^{\ell(A)} \dim \mathrm{Ext}_A^1(S_i, S_i)$.

(iii) $\ell(A) + \sum_{i=1}^{\ell(A)} \dim \mathrm{Ext}_A^1(S_i, S_i) \le k(A) \le \operatorname{tr} C_A$.

We prove that the codimension is Morita invariant and give an upper bound as well.

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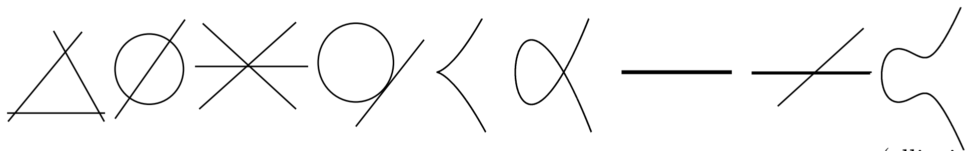
The defining relations of geometric algebras of Type EC

Ayako Itaba and Masaki Matsuno

This talk is a report of our paper [2]. Let k be an algebraically closed field of characteristic 0. An algebra A means a graded k-algebra finitely generated in degree 1. For a quadratic algebra $A = k\langle x_1, \cdots, x_n\rangle/I$ where $I = (f_1, \cdots, f_m)$ and f_j are homogeneous elements of degree 2, $\mathcal{V}(A) := \{(p, q) \in \mathbb{P}^{n-1} \times \mathbb{P}^{n-1} \mid f_j(p, q) = 0, j = 1, \cdots, m\}$ is called the point scheme of A.

Definition 1 ([3])**.** A quadratic algebra $A = k\langle x_1, \cdots, x_n\rangle/I$ is a geometric algebra if there exists a geometric pair (E, σ) where E is a projective variety in $\mathbb{P}^{n-1}$ and σ is an automorphism of E such that $\mathcal{V}(A) = \{(p, \sigma(p)) \mid p \in E\}$ and $I_2 = \{f \in k\langle x_1, \cdots, x_n\rangle_2 \mid f(p, \sigma(p)) = 0, \forall p \in E\}$.

If A is a geometric algebra, then A determines and is determined by the geometric pair (E, σ), so we write $A = \mathcal{A}(E, \sigma)$. A geometric algebra is related to an AS-regular algebra which is one of the main interests in noncommutative algebraic geometry. Artin-Tate-Van den Bergh [1] showed that there is a one-to-one correspondence between the set of 3-dimensional quadratic AS-regular algebras and the set of regular geometric pairs (E, σ) where E is $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$.



(elliptic curve)

In this talk, we study geometric algebras of Type EC, i.e., the algebras whose point schemes are elliptic curves in $\mathbb{P}^2$. We calculate defining relations of all geometric algebras of Type EC up to isomorphism of graded algebras by using the defining relations of 3-dimensional Sklyanin algebras. For $p \in E$, the automorphism σ_p defined by $\sigma_p(q) := p + q$ is called the translation by a point p. We will choose a suitable $\tau \in \operatorname{Aut}_k E$ such that $\operatorname{Aut}_k E = \{\sigma_p \tau^i \mid p \in E, i \in \mathbb{Z}_{|\tau|}\}$ where $|\tau|$ is the order of τ. We denote by $E[3]$ the set of 3-torsion points of E. The following is our main result.

Theorem 2 ([2])**.** *Let $p, q \in E \setminus E[3]$ and $i, j \in \mathbb{Z}_{|\tau|}$.*

(1) *Geometric algebras $\mathcal{A}(E, \sigma_p\tau^i)$ and $\mathcal{A}(E, \sigma_q\tau^j)$ of Type EC are isomorphic as graded algebras if and only if there exist $r \in E[3]$ and $l \in \mathbb{Z}_{|\tau|}$ such that $i = j$ and $q = \tau^l(p) + r - \tau^i(r)$.*

(2) *Geometric algebras $\mathcal{A}(E, \sigma_p\tau^i)$ and $\mathcal{A}(E, \sigma_q\tau^j)$ of Type EC are graded Morita equivalent if and only if there exist $r \in E[3]$ and $l \in \mathbb{Z}_{|\tau|}$ such that $p - \tau^{j-i}(p) \in E[3]$ and $q = \tau^l(p) + r$.*

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Quotients G/H is super-symmetry

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This is a report of my joint work with Yuta Takahashi, a PhD student at Tsukuba. Throughout we work over a field $\Bbbk$ of characteristic $\neq 2$.

Background. The trivial symmetry $v \otimes w \mapsto w \otimes v$, $V \otimes W \to W \otimes V$ on vector spaces is generalized to the super-symmetry

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v = \begin{cases} -w \otimes v & \text{if } v \text{ and } w \text{ are odd,} \\ w \otimes v & \text{otherwise.} \end{cases}$$

For the latter V and W are supposed to be *super-vector spaces*, or namely, vector spaces $V_0 \oplus V_1$, $W_0 \oplus W_1$ graded by $\mathbb{Z}/(2) = \{0, 1\}$, and v, w homogeneous elements of degree $|v|, |w|$. The super-vector spaces form a $\Bbbk$-linear abelian, symmetric tensor category with respect to the obvious tensor product and the super-symmetry above. Ordinary objects, such as Lie algebra or Hopf algebra, are generalized to super-objects, such as Lie super-algebra or Hopf super-algebra, in the category. My interest in the super-world, especially in algebraic super-groups, arose from the following striking result by P. Deligne (2002): if $\Bbbk$ is algebraically closed and $\operatorname{char} \Bbbk = 0$, a $\Bbbk$-linear abelian, rigid symmetric tensor category satisfying some mild assumption is realized as the category $\mathbb{G}$-$\mathsf{modules}$ of finite-dimensional super-modules over a certain affine algebraic super-group $\mathbb{G}$. Of course, before the result, there were already produced a huge number of fruitful results in super-geometry both by mathematicians and by physicists. But it was only around 2008 when the following basic question was explicitly posed by J. Brundan: *Given an affine algebraic super-group $\mathbb{G}$ and its algebraic super-subgroup $\mathbb{H}$, is the quotient (fppf-)sheaf $\mathbb{G}/\mathbb{H}$ a super-scheme?*

The results. In fact, the question was solved positively by Alexandr Zubkov and myself [J. Algebra 348 (2011), 135–170]. But this time Takahashi and I found a more acceptable proof of the result in the generalized situation that $\mathbb{G}$ may not be affine; the main point is to give an explicit description of $\mathbb{G}/\mathbb{H}$, see $(*)$ below. The description could be hopefully extended to Lie super-groups in an analytic context. The super-Grassmannians are typical, important examples of such quotients both in algebraic and analytic contexts.

Recall that *schemes* (over $\Bbbk$) are defined, from two viewpoints, as a sort of (i) topological spaces given structure sheaf of *commutative algebras* over $\Bbbk$, as well as, of (ii) functors defined on the category of *commutative algebras* over $\Bbbk$. It is not difficult to define *super-schemes*, replacing *commutative algebras* with super-commutative super-algebras. *Algebraic super-groups* are the group objects of algebraic super-schemes.

To show our result more explicitly, we let $\mathbb{H} \subset \mathbb{G}$ be as in Brundan's question above, concentrating on the affine case for simplicity. Then there are naturally associated (a) algebraic groups $H \subset G$ and (b) Lie super-algebras $\mathrm{Lie}(\mathbb{H}) \subset \mathrm{Lie}(\mathbb{G})$. (Here $\mathbb{G}$ is, from the viewpoint (ii), a group-valued functor, and G is the restricted functor defined on the category of the trivially graded super-algebras.) Note that (b) is an inclusion of left H-modules, where H acts by adjoint. Let $\mathsf{V}_{\mathbb{G}/\mathbb{H}} = \mathrm{Lie}(\mathbb{G})_1 / \mathrm{Lie}(\mathbb{H})_1$ be the associated quotient restricted to the odd component. A classical result tells us that the quotient sheaf G/H is a Noetherian scheme, and the quotient morphism $\pi : G \to G/H$ is affine, faithfully flat and finitely presented.

Theorem. *The quotient sheaf $\mathbb{G}/\mathbb{H}$ is a Noetherian super-scheme, whose underlying topological space $|\mathbb{G}/\mathbb{H}|$ is the same as $|G/H|$ of G/H. The quotient morphism $\mathbb{G} \to \mathbb{G}/\mathbb{H}$ has the same, desirable properties as π. Given an affine open subset $U \subset |G/H|$, we have a non-canonical isomorphism of super-algebras*

$$(*) \qquad \mathcal{O}_{\mathbb{G}/\mathbb{H}}(U) \simeq \wedge_{\mathcal{O}_{G/H}(U)}(\mathrm{Hom}_H(\mathsf{V}_{\mathbb{G}/\mathbb{H}},\ \pi_* \mathcal{O}_G(U))), \ \textit{the exterior algebra over } \mathcal{O}_{G/H}(U).$$

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