Resolution of DG-modules

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Motivation: projective dimension for DG-modules due to Yekutieli

A.Yekutieli introduced projective dimension $\mathrm{pd}_{\mathsf{R}}\,$M of DG-modules M over a DG-algebra R by generalizing the characterization of projective dimension ordinary modules by vanishing of Ext-group.
In this talk, we introduce a notion of projective resolution of DG-modules M, whose "length" gives $\mathrm{pd}_{\mathsf{R}}\,$M.

Let R be a ring and M an R-module. Then, the projective dimension is defined to be

$$\mathrm{pd}_R\, M := \inf\{\text{length of proj. resol.}\}.$$

Basic result is

$$\mathrm{pd}_R\, M = \sup\{d \mid \exists N \text{ s.t. } \mathrm{Ext}_R^d(M, N) \neq 0\}.$$

We reformulate this statement by using the derived category D(R).

Recall that

$$\mathrm{Ext}^{d}_{R}(M,N) = H^{d}(\mathbb{R}\mathrm{Hom}_{R}(M,N)).$$

Thus, introducing

$$D^{\leq d}(\mathbb{Z}) := \{X \in D(\mathbb{Z}) \mid H^{>d}(X) = 0\},$$

we obtain

$$\mathrm{pd}_{R} M \leq d \Leftrightarrow \mathbb{R}\mathrm{Hom}_{R}(M,N) \in D^{\leq d}(\mathbb{Z})$$

for all $N \in \mathrm{Mod}\, R$.

If we set $\mathsf{F} := \mathbb{R}\mathrm{Hom}_{\mathsf{R}}(\mathsf{M}, -)$, then the statement that

$$\mathrm{pd}_{\mathsf{R}}\,\mathsf{M} \le \mathsf{d} \Leftrightarrow \mathbb{R}\mathrm{Hom}_{\mathsf{R}}(\mathsf{M}, \mathsf{N}) \in \mathsf{D}^{\le \mathsf{n}}(\mathbb{Z})$$

for all $\mathsf{N} \in \mathrm{Mod}\,\mathsf{R}$, can be simply written

$$\mathrm{pd}_{\mathsf{R}}\,\mathsf{M} \le \mathsf{d} \Leftrightarrow \mathsf{F}(\mathrm{Mod}\,\mathsf{R}) \subset \mathsf{D}^{\le \mathsf{d}}(\mathbb{Z}).$$

We set

$$\mathsf{D}^{[m,n]}(R) := \{X \mid H^c(X) = 0, c \notin [m,n]\}.$$

Note $\mathsf{D}^{[0,0]}(R) = \operatorname{Mod} R$.

Recall $F = \mathbb{R}\mathrm{Hom}_R(M, -)$.

Lemma 1

$$\mathrm{pd}_R M \leq d \Leftrightarrow F(\mathsf{D}^{[m,n]}(R)) \subset \mathsf{D}^{[m,n+d]}(\mathbb{Z})$$

for any $m, n \in \mathbb{Z} \cap \{\pm\infty\}$.

In this talk, DG-algebra R is a non-positive DG-algebra, i.e., $\mathsf{H}^{>0}(\mathsf{R}) = 0$.
An important example of DGA is an ordinary ring.
A DGA R has its derived category D(R) and the same definition of $\mathsf{D}^{[m,n]}(\mathsf{R})$ works.
In the case R is an ordinary ring, these coincide with the previous categories.

If an algebra Λ has a silting object S in $D(\Lambda)$, then the DG-endomorphism algebra $R = \mathbb{R}\mathrm{End}\, S$ is a non-positive DGA and $D(\Lambda) \simeq D(R)$.
Thus, at the end of this talk, we obtain the notion of projective dimension of $M \in D(\Lambda)$ w.r.t. a silting object S and the global dimension of a silting object S.

Notation

Definition 2

Let $\mathbf{M} \in \mathbf{D}(\mathbf{R})$. Assume $\mathbf{M} \neq \mathbf{0}$.

1. $\sup \mathbf{M} := \sup\{n \in \mathbb{Z} \mid H^n(\mathbf{M}) \neq 0\}$
2. $\inf \mathbf{M} := \inf\{n \in \mathbb{Z} \mid H^n(\mathbf{M}) \neq 0\}$
3. $\operatorname{amp} \mathbf{M} := \sup \mathbf{M} - \inf \mathbf{M}$.

Let R be a DG-algebra, M a DG-R-module and $\mathbf{F} := \mathbb{R}\mathbf{Hom}_{\mathbf{R}}(\mathbf{M}, -)$.

Definition 3

1. $\mathbf{M}$ is said to have a projective concentration $[\mathbf{a}, \mathbf{b}]$ for $\mathbf{a} \leq \mathbf{b} \in \mathbb{Z}$, if
$$\mathbf{F}(\mathbf{D}^{[\mathbf{m},\mathbf{n}]}(\mathbf{R})) \subset \mathbf{D}^{[\mathbf{m}-\mathbf{b},\mathbf{n}-\mathbf{a}]}(\mathbf{R})$$
for any $\mathbf{m} \leq \mathbf{n} \in \mathbb{Z} \cup \{\pm\infty\}$.
2. A proj. conc. of $\mathbf{M}$ is called strict if there exists no smaller proj. conc. of $\mathbf{M}$.

Yekutieli's projective dimension for DG-modules (2/2)

Definition 4

Let $\mathbf{d} \in \mathbb{Z}$. We define $\mathrm{pd}_{\mathbf{R}}\,\mathbf{M} := \mathbf{d}$ if $\mathbf{M}$ has a strict proj. conc. $[\mathbf{a}, \mathbf{b}]$ such that $\mathbf{d} = \mathbf{b} - \mathbf{a}$.

We set $\mathrm{pd}_{\mathbf{R}}\,\mathbf{M} = \infty$ if $\mathbf{M}$ has no projective concentration.

Example 5

In the case $\mathbf{R}$ is an ordinary ring and $\mathbf{M}$ be an ordinary $\mathbf{R}$-module. Then $\mathrm{pd}_{\mathbf{R}}\,\mathbf{M}$ coincide the ordinary projective dimension.

Observations

Lemma 6

Let $\mathbf{M} \in \mathbf{D}(\mathbf{R})$ *and* $\mathbf{d} \in \mathbb{N}$.

1. *If* $\operatorname{pd}_{\mathbf{R}} \mathbf{M} < \infty$, *then* $\mathbf{M} \in \mathbf{D}^{-}(\mathbf{R})$, *i.e.* $\sup \mathbf{M} < \infty$ *or in other words,*
$$\mathrm{H}^{\mathbf{n}}(\mathbf{M}) = \mathbf{0} \text{ for } \mathbf{n} \gg \mathbf{0}$$
2. *Assume* $\mathbf{M} \in \mathbf{D}^{-}(\mathbf{R})$. *Then,* $\operatorname{pd}_{\mathbf{R}} \mathbf{M} = \mathbf{d}$ *if and only if* $\mathbf{M}$ *has a strict proj. conc.* $[\sup \mathbf{M} - \mathbf{d}, \sup \mathbf{M}]$.

We set $\mathcal{P} := \mathrm{Add}\, R \subset D(R)$.
This class plays a role of proj. modules.

Definition 7

Let $M \in D^{-}(R)$ and $s := \sup M$.
A morphism $f : P \to M$ is called sppj
if $P \in \mathcal{P}[-s]$ and
$H^{s}(f) : H^{s}(P) \to H^{s}(M)$ is surjective.

Sppj morphisms play roles of a surj. hom.
$f : P \twoheadrightarrow M$ with P projective.

Sup-projective (sppj) morphisms (2/3)

Proposition 8

Let $\mathbf{M} \in \mathbf{D}^{-}(\mathbf{R})$ and $\mathbf{f} : \mathbf{P} \to \mathbf{M}$ be a sppj morphism with the co-cone $\mathbf{N}$.

$$\mathbf{N} \to \mathbf{P} \xrightarrow{\mathbf{f}} \mathbf{M} \to \mathbf{N}[1] \quad (\text{exact}).$$

(1) If $\operatorname{pd}_{\mathbf{R}} \mathbf{M} \geq 1$, then

1. $\sup \mathbf{N} \leq \sup \mathbf{M}$.
2. $\operatorname{pd}_{\mathbf{R}} \mathbf{N} = \operatorname{pd}_{\mathbf{R}} \mathbf{M} - 1 - \sup \mathbf{M} + \sup \mathbf{N}$.
3. $\operatorname{pd}_{\mathbf{R}} \mathbf{N} < \operatorname{pd}_{\mathbf{R}} \mathbf{M}$.

Sup-projective (sppj) morphisms (3/3)

Proposition 9

$$\mathsf{N} \to \mathsf{P} \xrightarrow{\mathsf{f}} \mathsf{M} \to \mathsf{N}[1] \quad (\text{exact}).$$

(2) If $\mathrm{pd}_{\mathsf{R}}\, \mathsf{M} = 0$, *then* f *is split-epi.*
Thus, $\mathsf{M} \in \mathcal{P}[-\sup \mathsf{M}]$.

Corollary 10

$$\mathrm{pd}_{\mathsf{R}}\, \mathsf{M} = 0 \iff \mathsf{M} \in \mathcal{P}[-\sup \mathsf{M}].$$

Definition 11

A sppj resolution $\mathbf{P}_\bullet$ of $\mathbf{M}$ is a sequence $\{\mathcal{E}_i\}$ of exact triangles $\mathcal{E}_i$ for $\mathbf{i} \geq \mathbf{0}$ with $\mathbf{M}_0 := \mathbf{M}$

$$\mathcal{E}_i : \mathbf{M}_{i+1} \xrightarrow{g_{i+1}} \mathbf{P}_i \xrightarrow{f_i} \mathbf{M}_i$$

such that $\mathbf{f}_i$ is sppj. Set $\delta_i := \mathbf{g}_i \circ \mathbf{f}_i$.

Splicing $\mathcal{E}_i$'s we obtain

$$\cdots \to \mathbf{P}_i \xrightarrow{\delta_i} \mathbf{P}_{i-1} \to \cdots \to \mathbf{P}_1 \to \mathbf{P}_0 \xrightarrow{f_0} \mathbf{M}$$

projective dimension and sppj resolution (1/3)

Theorem 12

Let $\mathbf{M} \in \mathbf{D}^{-}(\mathbf{R}) \setminus \{0\}$ *and* $\mathbf{d} \in \mathbb{N}$ *a natural number. Then*
the following conditions are equivalent:

1. $\mathrm{pd}_{\mathbf{R}}\,\mathbf{M} = \mathbf{d}$.
2. $\mathbf{M}$ *has a sppj resolution* $\mathbf{P}_{\bullet}$ *of length* $\mathbf{e}$ *which satisfies the following properties.*
   (a) $\mathbf{d} = \mathbf{e} + \sup \mathbf{P}_0 - \sup \mathbf{P}_{\mathbf{e}}$.
   (b) $\delta_{\mathbf{e}}$ *is not a split-monomorphism.*

Theorem 12 (conti.): equivalent condition for $\operatorname{pd} \mathsf{M} = \mathsf{d}$

1. For any sppj resolution $P_\bullet$,
   there exists a natural number $e \in \mathbb{N}$
   which satisfying the following properties
   (a) $M_e \in \mathcal{P}[-\sup M_e]$.
   (b) $d = e + \sup P_0 - \sup M_e$.
   (c) g_e is not a split-monomorphism.

In this situation, we have a sppj resolution

$$M_e \xrightarrow{g_e} P_{e-1} \to \cdots \to P_1 \to P_0 \to M$$

Theorem 12 (conti.): equivalent condition for $\operatorname{pd} M = d$

1. Set $s := \sup M$ and $F = \mathbb{R}\mathrm{Hom}(M, -)$. Then

$$F(\operatorname{Mod} H^0(R)) \subset D^{[-s, d-s]}(R)$$

and $\exists N \in \operatorname{Mod} H^0$ such that $H^{d-s}(F(N)) \neq 0$.

2. d is the smallest number such that

$$M \in \mathcal{P}[-s] * \mathcal{P}[-s+1] * \cdots * \mathcal{P}[-s+d].$$

Let

$$\mathbf{D(R)_{fpd}} := \{\mathbf{M} \in \mathbf{D(R)} \mid \mathrm{pd}\,\mathbf{M} < \infty\}.$$

Corollary 13

$$\mathbf{D(R)}_{\mathsf{fpd}} = \mathbf{thick}\,\mathcal{P}.$$

Example (1/2)

Let $n \in \mathbb{N}$ and $M_{(n)} := R \oplus R[n]$.
Then, $\mathrm{pd}_R M_{(n)} = n$.
$M_{(n)}$ has a sppj resolution of length 1.

$$P_\bullet : R[n-1] \xrightarrow{0} R \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} M_{(n)}.$$

Compute the formula.

$$e + \sup P_0 - \sup P_e = 1 + 0 - (-(n-1)) = n$$

Example (2/2)

We have exact triangles

$$\mathsf{E_m}: \mathsf{M}_{(m-1)} \to \mathsf{R}^{\oplus 2} \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \mathsf{M}_{(m)}.$$

Since $\mathsf{M}_{(0)} = \mathsf{R}^{\oplus 2}$, splicing $\{\mathsf{E_m}\}_{m=1}^{n}$ we obtain a sppj resolution of length n

$$\mathsf{M}_{(0)} \to \mathsf{R}^{\oplus 2} \to \cdots \to \mathsf{R}^{\oplus 2} \to \mathsf{M}_{(n)}.$$

Compute the formula.

$$e + \sup P_0 - \sup P_e = n + 0 - 0 = n$$

By the same idea, Yekutieli also defined injective dimension $\mathrm{id}_{\mathsf{R}}\,\mathsf{M}$ of DG-modules. In the same way of sppj resolution, we can introduce the notion of inf-injective (ifij) resolution of DG-modules and prove results about injective dimension and ifij resolutions similar to those of sppj resolutions.

Theorem 14

For a non-pos. DGA $\mathbf{R}$*, the following numbers are the same.*

$$\sup\{\mathrm{pd}_{\mathbf{R}}\, \mathbf{M} - \mathbf{amp}\, \mathbf{M} \mid \mathbf{M} \in \mathbf{D}^{-}(\mathbf{R})\}$$
$$\sup\{\mathrm{pd}_{\mathbf{R}}\, \mathbf{M} \mid \mathbf{M} \in \mathbf{Mod}\, \mathbf{H}^{0}\}$$
$$\sup\{\mathrm{id}_{\mathbf{R}}\, \mathbf{M} - \mathbf{amp}\, \mathbf{M} \mid \mathbf{M} \in \mathbf{D}^{+}(\mathbf{R})\}$$
$$\sup\{\mathrm{id}_{\mathbf{R}}\, \mathbf{M} \mid \mathbf{M} \in \mathbf{Mod}\, \mathbf{H}^{0}\}$$

This common number is called the (right) global dimension of $\mathbf{R}$ *and is denoted as* $\mathbf{gldim}\, \mathbf{R}$.

Let R and S be derived equivalent non-pos. DG-algebras: $\mathsf{D}(\mathsf{R}) \simeq \mathsf{D}(\mathsf{S})$.

Theorem 15

The following assertions hold.

1. $\mathrm{pd}_{\mathsf{S}}\, \mathsf{R} < \infty$.
2. $\mathbf{gldim}\, \mathsf{S} \leq \mathbf{gldim}\, \mathsf{R} + \mathrm{pd}_{\mathsf{S}}\, \mathsf{R}$.
3. $\mathbf{gldim} \mathsf{R} < \infty$ *if and only if* $\mathbf{gldim}\, \mathsf{S} < \infty$.

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