An application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring

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Let A be an associative algebra over a commutative ring R. Let M be an A-bimodule. Assume that A is projective over R. Let $A^e := A \otimes_R A^{op}$ be the enveloping algebra of A. For A-bimodules A and M, we can regard them as left A^e-modules. We define the i-th *Hochschild cohomology group* $H^i(A, M)$ as $\mathrm{Ext}^i_{A^e}(A, M)$.

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$$C^i(A, M) := \begin{cases} \mathrm{Hom}_R(A^{\otimes i}, M) & (i \geq 0) \\ 0 & (i < 0) \end{cases}$$

and $d^i : C^i(A, M) \to C^{i+1}(A, M)$ $(i \geq 0)$ defined by

$$\begin{aligned} & d^i(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_{i+1}) \\ & := a_1 f(a_2 \otimes \cdots \otimes a_{i+1}) + \sum_{j=1}^{i} (-1)^j f(a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_{i+1}) \\ & \qquad + (-1)^{i+1} f(a_1 \otimes a_2 \otimes \cdots \otimes a_i) a_{i+1} \end{aligned}$$

for $f \in C^i(A, M)$. Here the tensor products are over R.

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Definition 2

Let X be a scheme. A subsheaf of $\mathcal{O}_X$-algebras $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ is said to be a *mold* of degree n on X if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on X.

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Proposition 1.2

The following contravariant functor is representable by a $\mathbb{Z}$-scheme $\mathrm{Mold}_{n,d}$.

$$\begin{array}{rcccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \,\middle|\, \mathcal{A} : \text{rank } d \text{ mold of degree } n \text{ on } X \right\} \end{array}$$

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Moreover, $\mathrm{Mold}_{n,d}$ is a closed subscheme of the Grassmann scheme $\mathrm{Grass}(d, n^2)$.

Example 3

In the case $n = 2$, we have

$$\begin{aligned} \mathrm{Mold}_{2,1} &= \mathrm{Spec}\mathbb{Z}, \\ \mathrm{Mold}_{2,2} &= \mathbb{P}^2_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,3} &= \mathbb{P}^1_{\mathbb{Z}}, \\ \mathrm{Mold}_{2,4} &= \mathrm{Spec}\mathbb{Z}. \end{aligned}$$

Example 4

Let $n = 3$. If $d = 1$ or $d \geq 6$, then

$$\begin{aligned}
\mathrm{Mold}_{3,1} &= \mathrm{Spec}\mathbb{Z}, \\
\mathrm{Mold}_{3,6} &= \mathrm{Flag} := \mathrm{GL}_3/\{(a_{ij}) \in \mathrm{GL}_3 \mid a_{ij} = 0 \text{ for } i > j\}, \\
\mathrm{Mold}_{3,7} &= \mathbb{P}^2_{\mathbb{Z}} \coprod \mathbb{P}^2_{\mathbb{Z}}, \\
\mathrm{Mold}_{3,8} &= \emptyset, \\
\mathrm{Mold}_{3,9} &= \mathrm{Spec}\mathbb{Z}.
\end{aligned}$$

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Theorem 6 (N- and T-)

The moduli $\mathrm{Mold}_{3,3}$ *has the following irreducible decomposition:*

$$\mathrm{Mold}_{3,3} = \overline{\mathrm{Mold}^{\mathrm{reg}}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_2}_{3,3}} \cup \overline{\mathrm{Mold}^{\mathrm{S}_3}_{3,3}}.$$

Today's talk

We talk about an application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring.

Zariski tangent space

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Let A be a k-subalgebra of $\mathrm{M}_n(k)$ over a field k. We define $\mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$ by

$$\begin{aligned}&\mathrm{Der}_k(A, \mathrm{M}_n(k)/A)\\ &\quad := \{f \in \mathrm{Hom}_k(A, \mathrm{M}_n(k)/A) \mid f(ab) = af(b) + f(a)b \text{ for } a, b \in A\}.\end{aligned}$$

Zariski tangent space

Proposition 1.3

Let $T_x\mathrm{Mold}_{n,d}$ *be the Zariski tangent space of* $\mathrm{Mold}_{n,d}$ *at* x*. There exists an isomorphism*

$$T_x\mathrm{Mold}_{n,d} \cong \mathrm{Der}_{k(x)}(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)).$$

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Proof. The Zariski tangent space $T_x\mathrm{Mold}_{n,d}$ consists of $k(x)[\epsilon]/(\epsilon^2)$-valued points of $\mathrm{Mold}_{n,d}$ mapping the closed point to x. We can easily check the statement. □

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For a k-subalgebra A of $\mathrm{M}_n(k)$, let us define
$d : \mathrm{M}_n(k) \to \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$ by

$$d(X)(a) := [X, a] = Xa - aX \mod A$$

for $X \in \mathrm{M}_n(k)$ and for $a \in A$. It is easy to check that
$d(X) \in \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$.

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Proposition 1.4

There exists an isomorphism

$$H^1(A, \mathrm{M}_n(k)/A) \cong \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)/\mathrm{Im}\, d.$$

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Proof. Let us consider the bar complex

$$0 \to C^0(A, \mathrm{M}_n(k)/A) \overset{d^0}{\to} C^1(A, \mathrm{M}_n(k)/A) \overset{d^1}{\to} C^2(A, \mathrm{M}_n(k)/A) \to \cdots .$$

Note that $\mathrm{Ker}\, d^1 = \mathrm{Der}_k(A, \mathrm{M}_n(k)/A) \supseteq \mathrm{Im}\, d^0 = \mathrm{Im}\, d$. Hence we have $H^1(A, \mathrm{M}_n(k)/A) \cong \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)/\mathrm{Im}\, d$. □

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The k-linear map $d : \mathrm{M}_n(k) \to \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$ induces a k-linear map $\overline{d} : \mathrm{M}_n(k)/A \to \mathrm{Der}_k(A, \mathrm{M}_n(k)/A)$.

Corollary 7

There exists the following exact sequence

$$0 \to N(A)/A \to \mathrm{M}_n(k)/A \xrightarrow{\overline{d}} \mathrm{Der}_k(A, \mathrm{M}_n(k)/A) \to H^1(A, \mathrm{M}_n(k)/A) \to 0.$$

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In particular,

$$\begin{aligned} &\dim_{k(x)} T_x \mathrm{Mold}_{n,d} \\ &\qquad = \dim_{k(x)} H^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) + n^2 - \dim_{k(x)} N(\mathcal{A}(x)) \end{aligned}$$

for $x \in \mathrm{Mold}_{n,d}$.

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Let $A \subseteq \mathrm{M}_n(R)$ be a rank d mold. In other words, A is an R-subalgebra of $\mathrm{M}_n(R)$ such that $\mathrm{M}_n(R)/A$ is projective and $\mathrm{rank}_R A = d$.

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If it always exists, the morphism $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ is (formally) smooth.

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$S(a_1), S(a_2), \dots, S(a_{n^2}) \in \mathrm{M}_n(\widetilde{R})$: lifts of $a_1, a_2, \dots, a_{n^2}$.
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Let us define an R-linear map $c' : A \otimes_R A \to \mathrm{M}_n(I) \cong \mathrm{M}_n(k) \otimes_k I$ by

$$c'\Big(\sum_{1 \le i,j \le d} r_{i,j} a_i \otimes a_j\Big) = \sum_{1 \le i,j \le d} s(r_{i,j})(S(a_i a_j) - S(a_i) S(a_j))$$

for $r_{i,j} \in R$.

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$S : \mathrm{M}_n(R) \to \mathrm{M}_n(\widetilde{R})$ is defined by $S(\sum_{i=1}^{n^2} r_i a_i) = \sum_{i=1}^{n^2} s(r_i)S(a_i)$.

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$$c'\Big(\sum_{1 \le i,j \le d} r_{i,j} a_i \otimes a_j\Big) = \sum_{1 \le i,j \le d} s(r_{i,j})(S(a_i a_j) - S(a_i)S(a_j))$$

for $r_{i,j} \in R$. Remark that I is a finite-dimensional k-vector space, since $mI = 0$ and I is a finitely generated ideal of R.

Smoothness

$s : R \to \widetilde{R}$ be a set theoretical section of $\pi : \widetilde{R} \to R$.
$A = \langle a_1, a_2, \dots, a_d \rangle \subseteq \mathrm{M}_n(R) = \oplus_{i=1}^{n^2} Ra_i$: a rank d mold.
$S(a_1), S(a_2), \dots, S(a_{n^2}) \in \mathrm{M}_n(\widetilde{R})$: lifts of $a_1, a_2, \dots, a_{n^2}$.
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Let $A_0 := A \otimes_R k \subseteq \mathrm{M}_n(k)$. Since $A = \oplus_{i=1}^{d} Ra_i$, we can write $A_0 = \oplus_{i=1}^{d} k\bar{a}_i$, where $\bar{a}_i := (a_i \mod m)$.

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It is easy to see that $c'' : A \otimes_R A \to (\mathrm{M}_n(k)/A_0) \otimes_k I$ goes through $A_0 \otimes_k A_0$.

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Then $c : A_0 \otimes_k A_0 \to (\mathrm{M}_n(k)/A_0) \otimes_k I$ is a cocycle in $C^2(A_0, (\mathrm{M}_n(k)/A_0) \otimes_k I)$.

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Proposition 1.6

The cohomology class $[c] \in H^2(A_0, (\mathrm{M}_n(k)/A_0) \otimes_k I)$ is independent from the choices of lifts s, S, and bases $a_1, \ldots, a_{n^2}$.

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Proposition 1.7

Let (R, m, k), $(\widetilde{R}, \widetilde{m}, k)$, I, and A_0 be as above. Let $A \in \mathrm{Mold}_{n,d}(R)$. There exists $\widetilde{A} \in \mathrm{Mold}_{n,d}(\widetilde{R})$ such that $\widetilde{A} \otimes_{\widetilde{R}} R = A$ if and only if the cohomology class $[c]$ is zero in $H^2(A_0, (\mathrm{M}_n(k)/A_0) \otimes_k I)$.

Smoothness

Theorem 8

Let $x \in \mathrm{Mold}_{n,d}$*. Let* $\mathcal{A}$ *be the universal mold on* $\mathrm{Mold}_{n,d}$*. Set* $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_{\mathrm{Mold}_{n,d}}} k(x)$*. If* $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$*, then the canonical morphism* $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ *is smooth at* x*.*

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Remark 1.8

Even if $H^2(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) \neq 0$, $\mathrm{Mold}_{n,d} \to \mathbb{Z}$ may be smooth at $x \in \mathrm{Mold}_{n,d}$.

Goal

Let's calculate $H^i(A, \mathrm{M}_n(R)/A)$!

Incidence algebra

Let Q be a finite quiver.

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Let RQ be the path algebra over a commutative ring R. We define the *arrow ideal* F as the two-sided ideal of RQ generated by the paths of positive length of Q.

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Definition 9

A two-sided ideal of I of RQ is called *admissible* if $F^n \subset I \subset F$ for a positive integer n and F/I is an R-free module which has an R-basis consisting of oriented paths.

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For an admissible ideal I, set $\Lambda = RQ/I$ and $r = F/I$. Denote by E the R-subalgebra of Λ generated by Q_0.

Incidence algebra

Proposition 1.9 (Cibils)

Let M be a Λ-bimodule.

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Let M be a Λ-bimodule. The Hochschild cohomology R-modules $H^i(\Lambda, M)$ are the cohomology groups of the complex of E-bimodules

$$0 \to M^E \xrightarrow{\delta^0} \mathrm{Hom}_{E^e}(r, M) \xrightarrow{\delta^1} \mathrm{Hom}_{E^e}(r \otimes_E r, M) \xrightarrow{\delta^2} \cdots$$

$$\cdots \xrightarrow{\delta^{i-1}} \mathrm{Hom}_{E^e}(r^{\otimes i}, M) \xrightarrow{\delta^i} \mathrm{Hom}_{E^e}(r^{\otimes i+1}, M) \xrightarrow{\delta^{i+1}} \cdots,$$

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where the tensor products are over E and

$$\begin{aligned} M^E &= \{m \in M \mid sm = ms \text{ for each } s \in Q_0\} \\ \delta^0(m)(x) &:= xm - mx \text{ for } m \in M^E \text{ and for } x \in r, \\ \delta^i(f)(x_1 \otimes \cdots \otimes x_{i+1}) &:= x_1 f(x_2 \otimes \cdots \otimes x_{i+1}) \\ &\quad + \sum_{j=1}^{i} (-1)^j f(x_1 \otimes \cdots \otimes x_j x_{j+1} \otimes \cdots \otimes x_{i+1}) \\ &\quad + (-1)^{i+1} f(x_1 \otimes \cdots \otimes x_i) x_{i+1}. \end{aligned}$$

Incidence algebra

Remark 1.10

Set $r^{\otimes 0} := E$. Then $\mathrm{Hom}_{E^e}(r^{\otimes 0}, M) = M^E$. Hence the complex above can be written by $\{\mathrm{Hom}_{E^e}(r^{\otimes n}, E), \delta^n\}$.

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Definition 10

Let Q be a finite quiver without oriented cycles. We say that Q is *ordered* if there exists no oriented path other than α joining $t(\alpha)$ to $h(\alpha)$ for each arrow $\alpha \in Q_1$.

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Let Q be an ordered quiver. Let I be the two-sided ideal of RQ generated by

$$\left\{ \gamma - \delta \in RQ \;\middle|\; \begin{array}{c} \gamma \text{ and } \delta \text{ are oriented paths with} \\ h(\gamma) = h(\delta) \text{ and } t(\gamma) = t(\delta) \end{array} \right\}.$$

We call $\Lambda = RQ/I$ an *incidence R-algebra*.

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For an ordered quiver Q, set $n := |Q_0|$. For $a, b \in Q_0$, we define $a \geq b$ if $a = b$ or there exists an oriented path α such that $t(\alpha) = a$ and $h(\alpha) = b$. Then $(Q_0, \geq)$ is a partially ordered set (i.e. poset). Let $\Lambda := RQ/I$ be the incidence algebra associated to Q. For $a \geq b$, let $e_{b,a}$ be the equivalence class of oriented paths from a to b in Λ. We can write $\Lambda = \oplus_{a \geq b} Re_{b,a}$. Fix a numbering on Q_0. By regarding $e_{b,a}$ as E_{ba}, Λ can be considered as an R-subalgebra of $\mathrm{M}_n(R) = \oplus_{a,b \in Q_0} RE_{ba}$, where E_{ba} is the matrix unit.

Incidence algebra

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Incidence algebra

Lemma 12

For $i \geq 0$, $\mathrm{Hom}_{E^e}(r^{\otimes i}, \mathrm{M}_n(R)/\Lambda) = 0$.

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Since $\mathrm{Hom}_{E^e}(Re_{s_i,s_0}, Re_{b,a}) \cong e_{s_i,s_i}(Re_{b,a})e_{s_0,s_0} = 0$,
$\mathrm{Hom}_{E^e}(r^{\otimes i}, \mathrm{M}_n(R)/\Lambda) = 0$. □

Incidence algebra

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Theorem 13

Let Q be an ordered quiver with $n = |Q_0|$. Let Λ be the incidence algebra associated to Q. Then $H^i(\Lambda, \mathrm{M}_n(R)/\Lambda) = 0$ for $i \geq 0$.

Example 14

Let us consider the following quiver Q

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n.$$

Incidence algebra

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Let $\Lambda = RQ/I$ be the incidence algebra associated to Q over a commutative ring R. Then $\Lambda = \oplus_{1 \leq i \leq j \leq n} Re_{i,j}$. We can regard Λ as the upper triangular matrix ring

$$\mathcal{B}_n(R) := \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ for } i > j\}.$$

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By Theorem 13,

$$H^i(\mathcal{B}_n(R), \mathrm{M}_n(R)/\mathcal{B}_n(R)) = 0$$

for $i \geq 0$.

Examples

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Definition 15

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Let $n_1, n_2, \ldots, n_s$ be positive integers with $\sum_{i=1}^{s} n_i = n$. We define the R-subalgebra $\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)$ of $\mathrm{M}_n(R)$ over a commutative ring R by

$$\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)$$
$$= \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ if } \sum_{k=1}^{t} n_k < i \leq \sum_{k=1}^{t+1} n_k \text{ and } j \leq \sum_{k=1}^{t} n_k\}.$$

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Proposition 1.11

Let R be a commutative ring. Let $\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)$ be as in Definition 15. Then $H^i(\mathcal{P}_{n_1,n_2,\ldots,n_s}(R), \mathrm{M}_n(R)/\mathcal{P}_{n_1,n_2,\ldots,n_s}(R)) = 0$ for $i \geq 0$.

Examples

Let R be a commutative ring, and let
$\mathrm{D}_n(R) := \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ for } i \neq j\} \subset \mathrm{M}_n(R)$.

Examples

Let R be a commutative ring, and let $\mathrm{D}_n(R) := \{(a_{ij}) \in \mathrm{M}_n(R) \mid a_{ij} = 0 \text{ for } i \neq j\} \subset \mathrm{M}_n(R)$. In other words, $\mathrm{D}_n(R)$ is the R-subalgebra of diagonal matrices in $\mathrm{M}_n(R)$.

Proposition 1.12

For $i \geq 0$, $H^i(\mathrm{D}_n(R), \mathrm{M}_n(R)/\mathrm{D}_n(R)) = 0$.

Examples

Definition 16

Let R be a commutative ring. We define $x \in \mathrm{M}_n(R)$ by

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Definition 16

Let R be a commutative ring. We define $x \in \mathrm{M}_n(R)$ by

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Let $\mathrm{J}_n(R)$ be the R-subalgebra of $\mathrm{M}_n(R)$ generated by x. Then $\mathrm{J}_n(R) \cong R[x]/(x^n)$ as R-algebras.

Examples

Proposition 1.13

Let $\mathrm{J}_n(R)$ *be as above. Then*

$$H^i(\mathrm{J}_n(R), \mathrm{M}_n(R)/\mathrm{J}_n(R)) = \begin{cases} R^{n-1} \oplus \mathrm{Ann}(n) & (i: \text{ even }) \\ R^{n-1} \oplus R/nR & (i: \text{ odd }), \end{cases}$$

where $\mathrm{Ann}(n) := \{a \in R \mid an = 0\}$.

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Corollary 17

For a field k,

$$H^i(\mathrm{J}_n(k), \mathrm{M}_n(k)/\mathrm{J}_n(k)) = \begin{cases} k^{n-1} & (\mathrm{ch}(k) \nmid n) \\ k^n & (\mathrm{ch}(k) \mid n). \end{cases}$$

Remark 1.14

Although $H^2(\mathrm{J}_n(k), \mathrm{M}_n(k)/\mathrm{J}_n(k)) \neq 0$, $\mathrm{Mold}_{n,n}$ is smooth at $\mathrm{J}_n(k)$.

Remark 1.14

Although $H^2(\mathrm{J}_n(k), \mathrm{M}_n(k)/\mathrm{J}_n(k)) \neq 0$, $\mathrm{Mold}_{n,n}$ is smooth at $\mathrm{J}_n(k)$. Indeed, $\mathrm{J}_n(k)$ is contained in $\mathrm{Mold}_{n,n}^{\mathrm{reg}} := \overline{\mathrm{Mold}_{n,n}^{\mathrm{D_n}}} \subseteq \mathrm{Mold}_{n,n}$ and $\mathrm{Mold}_{n,n}^{\mathrm{reg}}$ is smooth over $\mathbb{Z}$.

Examples

In the case $n = 3$:

$$S_2(k) := \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & c \\ 0 & 0 & b \end{pmatrix} \middle| \, a, b, c \in k \right\}$$

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Proposition 1.15

For $A = \mathrm{S}_2(k)$ *or* $\mathrm{S}_3(k)$*,*

$$H^i(A, \mathrm{M}_3(k)/A) = \begin{cases} k^2 & (i = 0) \\ 0 & (i > 0). \end{cases}$$

Examples

Theorem 18

For the non-commutative subalgebras part in $\mathrm{Mold}_{3,3}$, *we have*

$$\mathrm{Mold}_{3,3}^{\mathrm{non-comm}} = \mathrm{Mold}_{3,3}^{\mathrm{S}_2} \coprod \mathrm{Mold}_{3,3}^{\mathrm{S}_3}$$
$$\cong ((\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}) \setminus \Delta) \coprod ((\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}) \setminus \Delta),$$

where Δ *is the diagonal of* $\mathbb{P}^2_{\mathbb{Z}} \times \mathbb{P}^2_{\mathbb{Z}}$.

Examples

Corollary 19

Let (R, m, k) *be a local ring. Let* $A \subseteq \mathrm{M}_3(R)$ *be a rank 3 R-subalgebra of* $\mathrm{M}_3(R)$ *such that A and* $\mathrm{M}_3(R)/A$ *are R-projective and* $A \otimes_R k$ *is not commutative. Then there exists* $P \in \mathrm{GL}_3(R)$ *such that* $P^{-1}AP = \mathrm{S}_2(R)$ *or* $\mathrm{S}_3(R)$*.*

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Let (R, m, k) be a local ring. Let $A \subseteq \mathrm{M}_3(R)$ be a rank 3 R-subalgebra of $\mathrm{M}_3(R)$ such that A and $\mathrm{M}_3(R)/A$ are R-projective and $A \otimes_R k$ is not commutative. Then there exists $P \in \mathrm{GL}_3(R)$ such that $P^{-1}AP = \mathrm{S}_2(R)$ or $\mathrm{S}_3(R)$.

In other words, there exist distinct subline bunldes L_1, L_2 of R^3 or distinct rank 2 subbundles W_1, W_2 of R^3 such that

$$A = \langle \mathrm{Hom}_R(R^3/L_1, L_2) \rangle \subset \mathrm{End}_R(R^3) = \mathrm{M}_3(R)$$

or

$$A = \langle \mathrm{Hom}_R(R^3/W_1, W_2) \rangle \subset \mathrm{End}_R(R^3) = \mathrm{M}_3(R).$$