# On Projective Module with unique Maximal Submodule

The 51th Ring and Representation Theory Symposium
Okayama University of Science

Masahisa Sato

Aichi University & University of Yamanashi

13:10-13:40
September 21, 2018

The English version of this lecture was presented in ICRA2018 as follows;

On Projective Module with unique Maximal Submodule
18th International Conference
On Representations of Algebras (ICRA 2018)
Faculty of Civil Engineering, Czech Technical University
Prague (Czech Republic)
August 13-17, 2018.

This lecture will be done in Japanese.
So please ask your neighbors in case you do not understand Japanese.

# Ware's Problem

## Ware's Problem

Let $R$ be a ring and $P$ a projective right $R$-module with unique maximal submodule $L$, then $L$ is the largest maximal submodule of $P$.

R.Ware *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** (1971), 233-256.

## Original Problem:

$$\mathrm{End}_R(P_R) \text{ is local ring ?}$$

(i.e.) $P$ is completely indecomposable.

# Purpose

**Purpose**

Giving affirmative answer for this problem.

# Key facts

**Key facts**

W.K. Nicholson, M.F. Yousif, *Quasi-Frobenius Rings*, Cambridge University Press (2002).

F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, GTM **13**, Springer-Verlag (1992).

Key facts to solve Ware's problem:

(1) *Any projective module has a maximal submodule*.

This is equivalent to the following fact.

(2) *If $PJ(R) = P$ for a projective module $P$, then $P = 0$.*

# Remark

**Remark**

If $P$ is finitely generated projective $R$-module, then the following conditions are equivalent.

1. $P = eR$ for some local idempotent $e \in R$.
2. $\mathrm{End}_R(P)$ is local ring.
   (i.e.) $P$ is completely indecomposable.
3. $P$ has unique maximal submodule.
4. $P$ has the largest maximal submodule.

# Primitive rings and ideals

Definition 1

$R$ is right primitive ring

if $R$ has a faithful simple right $R$-module.

A two sided ideal $T$ of $R$ is a primitive right ideal

if $R/T$ is a primitive right ring.

An (right) annihilator of $S$ of $M_R$ is denoted by

$$\mathrm{Ann}_R(S) = \{r \in R \mid Sr = 0\}.$$

# Jacobson radical and primitive right ideal

1. **The Jacobson radical $J(R)$**
   1. The intersection of all maximal right ideals of $R$
   2. The intersection of all primitive right ideals of $R$

2. **What is a primitive right ideal $T$**
   1. A faithful simple right $R/T$-module $R/J$ is given by the form
      $$T = \mathrm{Ann}_R(R/J).$$
   2. $T$ is maximal on two sided ideals included in the above $J$.
   3. $T = \bigcap_{I \in \Gamma} I = \bigcap_{I \in \Delta} I$

      $\Gamma = \{$ maximal right ideals $I$ with $T \subset I$ $\}$
      $\Delta = \{$ maximal right ideals $I$ with $R/J \cong R/I$ $\}$

# Basic facts

We keep the follwing notations in this lecture.

$P$ is a projective right $R$-module with unique maximal submodule $L$

$J$ is a maximal right ideal such that $P/L \cong R/J$

$K = \mathrm{Ann}_R(R/J)$

# Basic facts

**Lemma 1**

1. *$K$ is maximal among two sided ideals included in $J$.*
2. $PK \subset L$.
3. $PI = P$ *for any maximal right ideal* $I$ *such that* $R/I \not\cong R/J$,
   $PI = P$ *for any primitive right ideal* $I \neq K$.
4. $T = \bigcap_{I_\gamma \in \Gamma} I_\gamma$, *then* $PT = \bigcap_{I_\gamma \in \Gamma} PI_\gamma$. *($\Gamma$ is a set of two sided ideals.)*
5. $\bigcap_{I \in \Gamma} PI = PJ(R)$. *($\Gamma$ is the set of primitive right ideals. )*
6. $L \supset PK = PJ(R)$.

# Example

In the R.Ware's problem, the assumption "projective" is necessary.

In fact, we give an example of a ring and a module with unique maximal submodule but not largest submodule.

# Example

## Example 1

$K$ is a field

$R$ is a $K$-algebra with bases $\{v_x \mid 0 \leq x \leq 1\}$

the multiplication $v_x \cdot v_y = v_{xy}$.

1. $R$ is a uniserial commutative ring.
2. The ideals of $R$:
   $J_i$ $(0 \leq i \leq 1)$ with $K$-bases $\{v_x \mid 0 \leq x < i\}$
   $\overline{J_i}$ with $K$-bases $\{v_x \mid 0 \leq x \leq i\}$. Closure ideal of $J_i$.

# Example

An $R$-module
$$M = (R \oplus J_1)/K\{(v_0, 0) - (0, v_0)\}$$
has unique maximal but not largest submodule.

In fact,

1. An $R$-module
$$L = (J_1 \oplus J_1)/K\{(v_0, 0) - (0, v_0)\}$$
is unique maximal submodule of $M$.
2. $$T = (J_{\frac{1}{2}} \oplus R)/K\{(v_0, 0) - (0, v_0)\}$$
is a submodule of $M$ not included in $L$.
3. There is no maximal submodule of $M$ which include $T$.

# Decomposition

### Proposition 2

*Let $M$ be a right $R$-module with unique maximal submodule $L$. Then*

1. *$M$ is indecomposable*
   *or*
2. *There are direct summands $M_1$ and $M_2$ such that*
   *$M = M_1 \oplus M_2$,*
   *$M_1$ has unique maximal submodule,*
   *$M_2$ does not have any maximal submodules.*

### Remark 3

By the former example, both cases in the above proposion happen.

# Decomposition proj.

Take a projective module $M = P$, then we have the following proposition.

### Corollary 4

*Let $P$ be a projective right $R$-module with unique maximal submodule $L$. Then*

1. *$P$ is indecomposable*
   *or*
2. *There are direct summands $P_1$ and $P_2$ such that*
   *$P = P_1 \oplus P_2$,*
   *$P_1$ has unique maximal submodule,*
   *$P_2$ does not have any maximal submodules.*

We show the second case does not happen.

# Generalized Nakayama-Azumaya Lemma

Theorem 5

*A nonzero projective module has a maximal submodule.*

Remark 6

In the proof of the above theorem, we use Axiom choice.
Also we can show this part by using Zorn's Lemma.

The above theorem is equivalent to the following property.

Theorem 7

*Let $P$ be a projective module. Then $PJ(R) = P$ implies $P = 0$.*

# Generalized Nakayama-Azumaya Lemma

Reviewing my proof.
What did I proved ?

The following theorem seems to be proved.

**Theorem 8 (Generalized Nakayama-Azumaya Lemma)**

*Let $M$ be a direct summand of a direct sum of finitely generated modules. Then $MJ(R) = M$ implies $M = 0$.*

# Indecomposablity

Theorem 9

*Let $R$ be a ring and $P$ a projective right $R$-module with unique maximal submodule $L$, then $P$ is indecomposable.*

# Example of Hinohara

An example of
infinitely generated indecomposable projective module $P$
is introduced in
S. Hinohara, *Projective modules II*, The sixth proceeding of Japan algebraic symposium (Homological algebra and its applications), Vol.**6** (1964), 24—28.

We review this example and we can show that
$PJ = P$ only for one maximal right ideal $J$.

# Example of Hinohara

## Example 2

$R$ : A commutative ring consisting of continuous real functions with the domain $[0, 1]$.

Maximal ideals : $\mathfrak{m}_x = \{f \in R \mid f(x) = 0\}$, $(x \in [0, 1])$

$P_x$ : An ideal consisting of $f \in R$ with $f(t) = 0$ for some neighborhood of $x$

Then

1. $P_x$ is infinitely generated indecomposable projective $R$-module by S.Hinohara.
2. It is countably generated by Kaplanski.
3. $P_x$ does not have a simple factor module isomorphic to $R/\mathfrak{m}_x$
4. $P_x$ has a simple factor module isomorphic to $R/\mathfrak{m}_y$ for any $y \neq x$ (since $P_x\mathfrak{m}_x = P_x$ and $P_x\mathfrak{m}_y \neq P_x$.)

# A reduced generator set

An indecomposabel projective module is countably generated
by I. Kaplansky, *Projective modules*, Ann. of Math. Vol.**68** (1958), 372–377.

**Definition 2**

Let $\{a_1, a_2, \cdots\}$ be a generator set of $P$.
This set is called a reduced generator set if it satisfies

$$a_{n+1} \notin a_1R + a_2R + \cdots + a_nR$$

for any $n$.

# A reduced generator set

**Lemma 10**

*Let $P$ be an indecomposable projective $R$-module. For any nonzero element $a \in P$, there is a reduced generator set such that $a = a_1$.*

**Lemma 11**

*Let $P$ be a projective right $R$-module with a maximal submodule $L$, then we can take a reduced generator set $\{a_1, a_2, \cdots\}$ of $P$ such that $a_1 \notin L$ and $a_i \in L$ $(i \geq 2)$.*

# Ware's problem is true

**Theorem 12**

*Let $R$ be a ring and $P$ a projective right $R$-module with unique maximal submodule $L$, then*
*$P$ is isomorphic to $eR$ for some local idempotent $e \in R$.*
*Particularly $L$ is the largest maximal submodule of $P$ and $L = eJ(R)$.*

This is proved by considering $M = a_1R + a_3R + \cdots$ and $P/M$ for some reduced generating set $\{a_1, a_2, \cdots\}$ such that $a_1 \notin L,\ a_i \in L\,(i > 1)$ and $M \neq P$.

# References

[1] F.W. Anderson, K.R. Fuller, *Rings and Categories of Modules*, GTM **13**, Springer-Verlag (1992).

[2] I.N. Hernstein, *Noncommutative Rings*, The Carus Mathematics Monographs 15, The Mathematical Association of America, (1968).

[3] S. Hinohara, *Projective modules II*, The sixth proceeding of Japan algebraic symposium (Homological algebra and its applications), Vol.**6** (1964), 24–28.

[4] I. Kaplansky, *Projective modules*, Ann. of Math. Vol.**68** (1958), 372–377.

[5] I. Kaplansky, *Fields and Rings*, The university of Chicago Press, Chixago (1969).

[6] W.K. Nicholson, M.F. Yousif, *Quasi-Frobenius Rings*, Cambridge University Press (2002).

[7] R. Ware, *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** (1971), 233–256.

Thank you for your attention !