Geometrical Construction of Quotients G/H in Super-symmetry

#### Akira Masuoka from Tsukuba Based on joint work with Yuta Takahashi arXiv:1808.05753

Okayama Science Univ., September 22, 2018

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Answer: Yes (M-Zubkov 2011): Yes. Moreover, the structure sheaf of the super-symmetric quotient G/H is...(M-Takahashi, this time)

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As a generalized symmetric tensor category we have the category (super-vector space) of super-vector spaces, based on which are defined *super-objects*, such as super-commutative super-algebra, super-Hopf algebra, super-Lie algebra.

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We have

(vector space)  $\subset$  (super-vector space)

so that ordinary objects A are precisely super-objects which are purely even,  $A = A_0$ .

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In what follows a super-algebra  $A = A_0 \oplus A_1$  is assumed to be super-commutative, so that A includes  $A_0$  as a central subalgebra, and ab = -ba,  $a^2 = 0$  for all  $a, b \in A_1$ .

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The *affine super-scheme* Spec A has  $Spec(A_0)$  as the underlying topological space, and the structure sheaf  $\mathcal{O}_{Spec A}$  is such that

$$\mathcal{O}_{\operatorname{Spec} A}(D(x)) = A \otimes_{\mathcal{A}_0} (\mathcal{A}_0)_x, \quad \mathcal{O}_{\operatorname{Spec} A, P} = A \otimes_{\mathcal{A}_0} (\mathcal{A}_0)_P,$$

where  $x \in A_0$ ,  $D(x) = \{x \notin P \in \text{Spec}(A_0)\}$  and  $P \in \text{Spec}(A_0)$ .

#### **Functorial viewpoint:**

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**Comparison Thm** (M-Zubkov 2011): Spec  $A \mapsto$  Sp A extends to a category equivalence

$$({\sf super-scheme}) pprox egin{pmatrix} {\sf functorial} \ {\sf super-scheme} \end{pmatrix}$$

which assigns to X, the k-functor Mor(Spec(-), X) represented by X.

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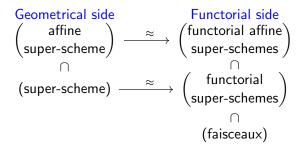
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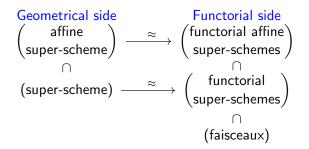
**Def (Grothendieck):** A *faisceau* (or  $\Bbbk$ -*sheaf*) is a  $\Bbbk$ -functor F which preserve finite products, and turns every equalizer diagram

$$R \to S \rightrightarrows S \otimes_R S$$

associated with an fppf (= faith. flat and finitely presented) map  $R \rightarrow S$  of super-algebras into the equalizer diagram of sets

$$F(R) \to F(S) \rightrightarrows F(S \otimes_R S).$$

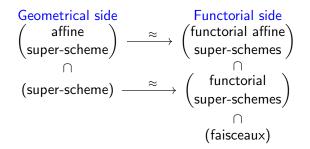




**Fact:** Given a k-functor X there uniquely exists a faisceau  $\tilde{X}$  equipped with a morphism  $\iota: X \to \tilde{X}$ , such that the induced map

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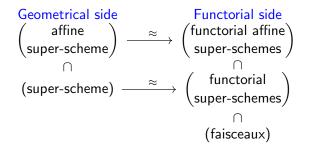


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**Appl:** Let  $\tilde{G/H}$  be the faisceau associated with the coset functor  $R \mapsto G(R)/H(R)$ . If this  $\tilde{G/H}$  happens to be a super-scheme, that is the quotient super-scheme G/H.

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represented by the super-Hopf algebra

$$A = \mathbb{k} \begin{bmatrix} x_{ij}, y_{k\ell}, \det(X)^{-1}, \det(Y)^{-1} \end{bmatrix} \otimes \wedge (p_{i\ell}, q_{kj});$$
$$\begin{pmatrix} X & P \\ Q & Y \end{pmatrix} = \begin{pmatrix} x_{ij} & p_{i\ell} \\ q_{kj} & y_{k\ell} \end{pmatrix}, \ x_{ij}, y_{k\ell} \text{ even}, \ p_{i\ell}, q_{kj} \text{ odd variables}$$
$$\Delta \begin{pmatrix} X & P \\ Q & Y \end{pmatrix} = \begin{pmatrix} X & P \\ Q & Y \end{pmatrix} \otimes \begin{pmatrix} X & P \\ Q & Y \end{pmatrix}, \ \varepsilon \begin{pmatrix} X & P \\ Q & Y \end{pmatrix} = \begin{pmatrix} I & O \\ O & I \end{pmatrix}$$

Alternative description of super-Lie algebras.

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$$R \mapsto G_0(R_0) \ltimes (V \otimes R_1)$$

so that  $(v \otimes a)(w \otimes b) = \exp([v, w] \otimes ab)(w \otimes b)(v \otimes a).$ 

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$$\mathsf{Gr}(r|s,m|n)(R) = ig\{ M <\!\!\!\oplus R^{m|n} \mid M_P \simeq R_P^{r|s} \;\; orall P \in \mathsf{Spec}(R_0) ig\}.$$

This is a super-scheme, covered by affine super-schemes  $F_W$ ,

$$F_W(R) = ig\{ M \mid M \oplus (W \otimes R) = R^{m|n} ig\},$$

where  $W \subset \mathbb{k}^{m|n}$  are sub-super-vector spaces  $\simeq \mathbb{k}^{m-r|n-s}$ .

Let  $r \leq m, s \leq n$  be integers  $\geq 0$ . Let  $\mathbb{k}^{m|n} := \mathbb{k}^m \oplus \mathbb{k}[1]^n$ . Given  $R \in (\text{super-algebra})$ , let  $R^{m|n} := \mathbb{k}^{m|n} \otimes R = R^m \oplus R[1]^n$ . The super-Grassmanian Gr(r|s, m|n) is the  $\mathbb{k}$ -functor

$$\mathsf{Gr}(r|s,m|n)(R) = \left\{ M \triangleleft R^{m|n} \mid M_P \simeq R_P^{r|s} \;\; \forall P \in \mathsf{Spec}(R_0) 
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$$\operatorname{GL}(m|n)(R) o \operatorname{Gr}(r|s,m|n)(R), \ g \mapsto g(U \otimes R)$$

gives rise to

$$\mathsf{GL}(m|n)\widetilde{/}\mathsf{Stab}_{\mathsf{GL}(m|n)}(U) \overset{\simeq}{\longrightarrow} \mathsf{Gr}(r|s,m|n).$$

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Let  $U \subset G_0/H_0$  be affine open, and regard  $\pi^{-1}(U)$  ( $\subset |G_0| = |G|$ ) as an open sub-super-scheme of G. The key is to construct a nice H-equivariant affine super-scheme  $X_U$  together with an H-equiv. open embedding

$$X_U \xrightarrow{\simeq} \pi^{-1}(U) \subset G.$$

$$X_U = \operatorname{Spec}((\mathcal{O}_{G_0}(\pi^{-1}(U)) \otimes \wedge(Z)) \times^{H_0} H),$$

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The resulting super-scheme is the quotient G/H, and represents the faisceau  $\tilde{G/H}$ . It has additional desirable properties including: an open subset  $U \subset |G/H| (= |G_0/H_0|)$  is affine in  $G/H \Leftrightarrow$  it is affine in  $G_0/H_0$ .

# Thanks, also to the organizers!