Geometrical Construction of
Quotients G/H in Super-symmetry

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Based on joint work with Yuta Takahashi
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As a generalized symmetric tensor category we have the category (super-vector space) of super-vector spaces, based on which are defined *super-objects*, such as super-commutative super-algebra, super-Hopf algebra, super-Lie algebra.

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We have

$$\text{(vector space)} \subset \text{(super-vector space)}$$

so that ordinary objects A are precisely super-objects which are purely even, $A = A_0$.

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The *affine super-scheme* $\operatorname{Spec} A$ has $\operatorname{Spec}(A_0)$ as the underlying topological space, and the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ is such that

$$\mathcal{O}_{\operatorname{Spec} A}(D(x)) = A \otimes_{A_0} (A_0)_x, \quad \mathcal{O}_{\operatorname{Spec} A, P} = A \otimes_{A_0} (A_0)_P,$$

where $x \in A_0$, $D(x) = \{x \notin P \in \operatorname{Spec}(A_0)\}$ and $P \in \operatorname{Spec}(A_0)$.

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Comparison Thm (M-Zubkov 2011): $\mathrm{Spec}\, A \mapsto \mathrm{Sp}\, A$ extends to a category equivalence

$$(\text{super-scheme}) \approx \begin{pmatrix} \text{functorial} \\ \text{super-scheme} \end{pmatrix}$$

which assigns to X, the $\mathbb{k}$-functor $\mathrm{Mor}(\mathrm{Spec}(-), X)$ *represented by* X.

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Def (Grothendieck): A *faisceau* (or $\Bbbk$-*sheaf*) is a $\Bbbk$-functor F which preserve finite products, and turns every equalizer diagram

$$R \to S \rightrightarrows S \otimes_R S$$

associated with an fppf (= faith. flat and finitely presented) map $R \to S$ of super-algebras into the equalizer diagram of sets

$$F(R) \to F(S) \rightrightarrows F(S \otimes_R S).$$

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Fact: Given a $\Bbbk$-functor X there uniquely exists a faisceau $\tilde{X}$ equipped with a morphism $\iota : X \to \tilde{X}$, such that the induced map

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is a bijection for every faisceau Y.

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Appl: Let $G\tilde{/}H$ be the faisceau associated with the coset functor $R \mapsto G(R)/H(R)$.

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Appl: Let $G\tilde{/}H$ be the faisceau associated with the coset functor $R \mapsto G(R)/H(R)$. If this $G\tilde{/}H$ happens to be a super-scheme, that is the quotient super-scheme G/H.

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represented by the super-Hopf algebra

$$A = \Bbbk\big[x_{ij}, y_{k\ell}, \det(X)^{-1}, \det(Y)^{-1}\big] \otimes \wedge(p_{i\ell}, q_{kj});$$

$$\begin{pmatrix} X & P \\ Q & Y \end{pmatrix} = \begin{pmatrix} x_{ij} & p_{i\ell} \\ q_{kj} & y_{k\ell} \end{pmatrix}, \ x_{ij}, y_{k\ell} \text{ even},\ p_{i\ell}, q_{kj} \text{ odd variables}$$

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$$R \mapsto G_0(R_0) \ltimes (V \otimes R_1)$$

so that $(v \otimes a)(w \otimes b) = \exp([v, w] \otimes ab)(w \otimes b)(v \otimes a)$.

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$$\mathrm{Gr}(r|s, m|n)(R) = \left\{ M \mathrel{<\!\!\!\oplus} R^{m|n} \mid M_P \simeq R_P^{r|s} \;\; \forall P \in \mathrm{Spec}(R_0) \right\}.$$

This is a super-scheme, covered by affine super-schemes F_W,

$$F_W(R) = \left\{ M \mid M \oplus (W \otimes R) = R^{m|n} \right\},$$

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Fix a sub-super-vector space $U \subset \mathbb{k}^{m|n}$ s.t. $U \simeq \mathbb{k}^{r|s}$. Then

$$\mathrm{GL}(m|n)(R) \to \mathrm{Gr}(r|s, m|n)(R), \; g \mapsto g(U \otimes R)$$

gives rise to

$$\mathrm{GL}(m|n) \tilde{/} \mathrm{Stab}_{\mathrm{GL}(m|n)}(U) \xrightarrow{\simeq} \mathrm{Gr}(r|s, m|n).$$

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Classical result (Grothendieck): There exists the quotient scheme G_0/H_0, which is Noetherian and represents the faisceau $G_0 \tilde{/} H_0$. The morphism $\pi : G_0 \to G_0/H_0$ is affine and fppf.

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Let $U \subset G_0/H_0$ be affine open, and regard $\pi^{-1}(U)\,(\subset |G_0| = |G|)$ as an open sub-super-scheme of G. The key is to construct a nice H-equivariant affine super-scheme X_U together with an H-equiv. open embedding

$$X_U \xrightarrow{\simeq} \pi^{-1}(U) \subset G.$$

Using Hopf-algebra techniques we construct X_U so as

$$X_U = \mathrm{Spec}\big((\mathcal{O}_{G_0}(\pi^{-1}(U)) \otimes \wedge(Z)) \times^{H_0} H\big),$$

where $Z = (\mathrm{Lie}(G)_1/\mathrm{Lie}(H)_1)^*$.

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Thm (M-Takahashi): The affine super-schemes $X_U\tilde{/}H$, where U ranges over the affine open subschemes of G_0/H_0, are glued into a Noetherian super-scheme with the underlying topological space $|G_0/H_0|$, whose struc. sheaf is locally isomorphic to

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The resulting super-scheme is the quotient G/H, and represents the faisceau $G \tilde{/} H$. It has additional desirable properties including: an open subset $U \subset |G/H| \, (= |G_0/H_0|)$ is affine in $G/H \Leftrightarrow$ it is affine in G_0/H_0.

Thanks, also to the organizers!