ON BALANCED AUSLANDER-DLAB-RINGEL ALGEBRAS

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Abstract. In this note, we give a sufficient condition for an Auslander–Dlab–Ringel algebra to be balanced quasi-hereditary.

1. Preliminaries

Throughout this note, $\mathbf{k}$ is a field and $\mathbb{D} := \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k})$. Let Λ be a basic finite dimensional $\mathbf{k}$-algebra with a complete set $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents. For $i \in I := \{1, 2, \dots, n\}$, let $P(i) = e_i\Lambda$, $E(i) = \mathbb{D}(\Lambda e_i)$ and $S(i) = \mathrm{top}P(i) = \mathrm{soc}E(i)$.

1.1. **Quasi-hereditary algebras.**

In this subsection, we recall the definitions of quasi-hereditary algebras and strongly quasi-hereditary algebras. For details, we refer to [2, 9].

Definition 1. Let $\leq$ be a partial order on I.

(1) A pair $(\Lambda, \leq)$ is called a *quasi-hereditary algebra* if there exist Λ-modules $\Delta(i)$ $(i \in I)$ such that
  - (a) there is a surjection $\Delta(i) \to S(i)$ with kernel having composition factors $S(j)$ with $j < i$,
  - (b) there is a surjection $P(i) \to \Delta(i)$ with kernel being filtered by $\Delta(j)$ with $j > i$.

  We call $\Delta(i)$ the *standard* module with respect to i.

(2) A quasi-hereditary algebra $(\Lambda, \leq)$ is said to be *right-strongly* if the projective dimension of each standard Λ-module is at most one. A quasi-hereditary algebra $(\Lambda, \leq)$ is said to be *left-strongly* if $(\Lambda^{\mathrm{op}}, \leq)$ is right-strongly.

(3) A quasi-hereditary algebra $(\Lambda, \leq)$ is said to be *strongly* if $(\Lambda, \leq)$ is right-strongly and left-strongly.

Note that $(\Lambda, \leq)$ is a quasi-hereditary algebra if and only if there exist Λ-modules $\nabla(i)$ $(i \in I)$ such that

- there is an injection $S(i) \to \nabla(i)$ with cokernel having composition factors $S(j)$ with $j < i$,
- there is an injection $\nabla(i) \to E(i)$ with cokernel being filtered by $\nabla(j)$ with $j > i$.

We call $\nabla(i)$ the *costandard* module with respect to i.

By [8], a quasi-hereditary algebra Λ has a basic tilting-cotilting Λ-module T, which is a direct sum of all indecomposable Ext-injective objects in the full subcategory of $\mathsf{mod}\Lambda$ whose objects are the modules that admit a Δ-filtration, i.e., a filtration whose subquotients are standard modules. Moreover, $R(\Lambda) := \mathrm{End}_{\Lambda}(T)^{\mathrm{op}}$ is a quasi-hereditary

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algebra with respect to the opposite order of $\leq$ and $R(R(\Lambda)) \cong \Lambda$. We call $R(\Lambda)$ the Ringel dual of $(\Lambda, \leq)$.

1.2. **Auslander–Dlab–Ringel algebras.** In this subsection, we recall the definition and basic properties of Auslander–Dlab–Ringel algebras. Let Λ be a basic finite dimensional $\mathbf{k}$-algebra and J_Λ the Jacobson radical of Λ. Put

$$G := \bigoplus_{i \in I} \bigoplus_{j=1}^{l_i} P(i)/P(i)J_\Lambda^j,$$

where l_i is the Loewy length of the indecomposable projective Λ-module $P(i)$. We call the endomorphism algebra $A := \mathrm{End}_\Lambda(G)$ the *Auslander–Dlab–Ringel* (*ADR*) *algebra* of Λ. Then the complete set of isomorphism classes of indecomposable projective A-modules are given by

$$\{P(i,j) := \mathrm{Hom}_\Lambda(G, P(i)/P(i)J_\Lambda^j) \mid i \in I,\ 1 \leq j \leq l_i\}.$$

Let

$$\mathcal{S}_A := \{(i,j) \mid i \in I,\ 1 \leq j \leq l_i\}.$$

For $(i,j), (k,l) \in \mathcal{S}_A$, we write $(i,j) \trianglelefteq (k,l)$ if $j \geq l$. Then $\trianglelefteq$ gives a partial order on $\mathcal{S}_A$.

Auslander [1] shows the global dimension of A is finite, and Dlab–Ringel [4] proves that $(A, \trianglelefteq)$ is a quasi-hereditary algebra. Moreover, we have the following result.

Proposition 2. [3, 10] *Let Λ be a finite dimensional algebra. Then the ADR algebra A of Λ is a left-strongly quasi-hereditary algebra. Moreover, the following statements are equivalent.*

(1) $(A, \trianglelefteq)$ *is a strongly quasi-hereditary algebra.*
(2) $\mathrm{gldim} A = 2$.
(3) $J_\Lambda \in \mathsf{add} G$.

1.3. **Koszul algebras.** Let us recall the definition of Koszul algebras and Koszul duals, originated from [7]. We will mostly follow the convention of [5]. We call a $\mathbb{Z}$-graded $\mathbf{k}$-algebra $\Gamma = \oplus_{n \in \mathbb{Z}} \Gamma_n$ is called a *positively graded algebra* if the following three conditions are satisfied.

- Γ_0 is semisimple.
- $\dim_K \Gamma_n < \infty$ for all n.
- $\Gamma_n = 0$ for all $n < 0$.

We denote by $\langle 1 \rangle$ the grading shift (endo-)functor on the category of graded Γ-modules. Let us assume for simplicity that all algebras we consider are finite-dimensional. Then every (ungraded) projective Γ-module P can be canonically graded with its top concentrated in degree 0. In particular, every (ungraded) simple Γ-module S is also canonically graded. By abuse of notation, we use the same symbol for the graded lifts of these modules.

Definition 3. A positively graded algebra Γ is called a *Koszul algebra* if each simple module S (concentrated in degree 0) admits a projective resolution in the category of graded Γ-modules of the form

$$\cdots \to P_j\langle -j \rangle \to \cdots \to P_1\langle -1 \rangle \to P_0 \to S \to 0,$$

where P_j are projective Γ-modules with the canonical grading.

When Γ is a Koszul algebra, then the Yoneda algebra $E(\Gamma) := \mathrm{Ext}^{\bullet}_{\Gamma}(\Gamma_0, \Gamma_0)$ is a Koszul algebra and $E(E(\Gamma)) \cong \Gamma$ as positively graded algebras.

1.4. **Balanced quasi-hereditary algebras.** For a positively graded quasi-hereditary algebra Γ, one can ask the following question.

Do the operations of taking the Ringel dual and taking the Koszul dual commute?

A sufficient condition for these two dual to commute is given in [5, 6], and quasi-hereditary algebras satisfying such a condition are called balanced; let us recall its definition now.

Let Γ be a positively graded quasi-hereditary algebra. As in the case of projective and simple modules, one can show that the standard modules, the costandard modules, and the characteristic tilting module admit a canonical grading.

Definition 4. Let Γ be a positively graded quasi-hereditary algebra and let T be its (canonically graded) characteristic tilting module. Then Γ is called a *balanced quasi-hereditary algebra* if each (canonically graded) standard module Δ admits an exact sequence in the category of graded modules of the form:

$$0 \to \Delta \to T^0 \to T^1\langle 1\rangle \to \cdots \to T^i\langle i\rangle \to \cdots$$

with $T^j \in \mathsf{add}T$ ($^\forall j \geq 0$), and each (canonically graded) costandard module ∇ admits an exact sequence in the category graded modules of the form:

$$\cdots \to T_i\langle -i\rangle \to \cdots \to T_1\langle -1\rangle \to T_0 \to \nabla \to 0$$

with $T_j \in \mathsf{add}T$ ($^\forall j \geq 0$).

Let us state the result of [5] more precisely.

Theorem 5. [5] *Let Γ be a balanced quasi-hereditary algebra. Then the following statements hold.*

(1) Γ *is a Koszul algebra.*
(2) $R(\Gamma)$ *and* $E(\Gamma)$ *are also balanced quasi-hereditary algebras and hence Koszul.*
(3) $E(R(\Gamma)) \cong R(E(\Gamma))$.

2. Main results

In this section, we give a sufficient condition for an ADR algebra to be balanced quasi-hereditary. Let $\mathbf{k}$ be an algebraically closed field. Let Λ be a basic connected finite dimensional $\mathbf{k}$-algebra with a complete set $\{e_1, e_2, \dots, e_n\}$ of primitive orthogonal idempotents.

Let $Q = (Q_0, Q_1)$ be the Gabriel quiver of Λ. Define a new quiver $Q^{ADR} = (Q'_0, Q'_1)$ as follows.

$$\begin{aligned} Q'_0 &:= \{(i,k) \mid i \in Q_0, 1 \leq k \leq \ell_i\},\\ Q'_1 &:= \{\alpha_{i,k} : (i,k) \to (i,k+1) \mid i \in Q_0, 1 \leq k < \ell_i\}\\ &\quad \sqcup \{\beta_{i,k} : (i,k) \to (j,k-1) \mid (i \to j) \in Q_1, 1 < k \leq \ell_i\}. \end{aligned}$$

Let I^{ADR} be a two-sided ideal of the path algebra $\mathbf{k}Q^{ADR}$ generated by the following two relations: For each arrow $(i \to j) \in Q_1$, $\alpha_{i,1}\beta_{i,2}$ and $\alpha_{i,k}\beta_{i,k+1} - \beta_{i,k}\alpha_{j,k-1}$ $(1 < k < l_i)$.

A module is said to be *rigid* if its radical series coincides with its socle series. A direct sum of indecomposable rigid modules is called a *semirigid module.*

Lemma 6. *Let A be the ADR algebra of Λ. Assume Λ is a semirigid Λ-module with $J_\Lambda \in \mathsf{add}G$. Then $A \cong \mathbf{k}Q^{ADR}/I^{ADR}$ and is positively graded quasi-hereditary with grading given by path length.*

Using Lemma 6, we can show the following key lemma.

Lemma 7. *There exist exact sequences in the category of graded modules*

$$0 \to \Delta(i,k) \to T(i,k) \to \oplus_{(j\to i)\in Q_1} T(j,k+1)\langle 1\rangle \to 0,$$
$$0 \to T(i,k+1)\langle -1\rangle \to T(i,k) \to \nabla(i,k) \to 0.$$

This allows us to deduce the desired sufficient condition.

Theorem 8. *Let Λ be a finite dimensional algebra and let A be its ADR algebra. Assume that A is strongly quasi-hereditary. Then the following statements are equivalent.*

(1) *A is a balanced quasi-hereditary algebra.*
(2) *A is a Koszul algebra.*
(3) *Λ is a semirigid Λ-module.*

Proof. (1)⇒(2): This follows from Proposition 5(1).

(2)⇒(3): We can show that if Λ is not semirigid, then A has a non-homogeneous relation. This implies A is not quadratic, and hence not Koszul.

(3)⇒(1): This follows from Lemma 7. □

References

[1] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College, 1971.

[2] E. Cline, B. Parshall, L. Scott, *Finite-dimensional algebras and highest weight categories*, J. Reine Angew. Math. **391** (1988), 85–99.

[3] T. Conde, *The quasihereditary structure of the Auslander-Dlab-Ringel algebra*, J. Algebra **460** (2016), 181–202.

[4] V. Dlab, C. M. Ringel, *Every semiprimary ring is the endomorphism ring of a projective module over a quasihereditary ring*, Proc. Amer. Math. Soc. **107** (1989), no. 1, 1–5.

[5] V. Mazorchuk, *Koszul duality for stratified algebras. I. Balanced quasi-hereditary algebras*, Manuscripta Math. **131** (2010), no. 1–2, 1–10.

[6] V. Mazorchuk, S. Ovsienko, *A pairing in homology and the category of linear complexes of tilting modules for a quasi-hereditary algebra. With an appendix by Catharina Stroppel*, J. Math. Kyoto Univ. **45** (2005), no. 4, 711–741.

[7] S. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39–60.

[8] C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. **208** (1991), no. 2, 209–223.

[9] C. M. Ringel, *Iyama's finiteness theorem via strongly quasi-hereditary algebras*, J. Pure Appl. Algebra **214** (2010), no. 9, 1687–1692.

[10] M. Tsukamoto, *On an upper bound for the global dimension of Auslander–Dlab–Ringel algebras*, Arch. Math. (Basel) **112** (2019), no. 1, 41–51.

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