ON THE WEAKLY IWANAGA–GORENSTEIN PROPERTY OF GENDO ALGEBRAS

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Abstract. We explore when an algebra is weakly Iwanaga–Gorenstein and show that the endomorphism algebra of a generator over a representation-finite algebra is weakly Iwanaga–Gorenstein.

1. Introduction

We study the weakly Iwanaga–Gorenstein property of algebras, which was introduced by Ringel–Zhang [RZ] as a generalization of Iwanaga–Gorenstein algebras. In particular, we deal with gendo algebras, that is, the ENDOmorphism algebras of Generators [FK]. These often have nice properties. For example, Morita's theorem says that the endomorphism algebra of a progenerator admits the same module category as the original algebra. Auslander introduced the notion of representation dimensions and Auslander algebras, which are also examples of gendo algebras. As is well-known, they give interesting interaction between representation theoretic properties and homological properties.

We ask when a gendo algebra is weakly Iwanaga–Gorenstein.

Theorem 1. *Let Λ be a finite dimensional algebra over a field and M a finite dimensional Λ-module. If Λ is representation-finite, then the gendo algebra $\mathrm{End}_\Lambda(\Lambda \oplus M)$ is wealy Iwanaga–Gorenstein.*

2. Result

Throughout this note, algebras are always assumed to be finite dimensional over an algebraically closed field K. By a module, we mean a finitely generated right module. For an algebra Λ, we denote by $\mathsf{mod}\,\Lambda$ ($\mathsf{proj}\,\Lambda$) the category of (projective) modules over Λ. One defines a full subcategory ${}^\perp\Lambda$ of $\mathsf{mod}\,\Lambda$ as follows:

$$^\perp\Lambda := \{X \in \mathsf{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(X, \Lambda) = 0 \text{ for any } i > 0\}.$$

We say that a module X is *Gorenstein-projective (abbr. GP)* if there is an exact sequence $\cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} \cdots$ such that $X_0 = X$ and the inclusion $X_i \to P^i$ is a left $\mathsf{proj}\,\Lambda$-approximation of X_i; that is, any homomorphism from X_i to a projective module factors through the inclusion. Here, $X_i := \mathrm{Ker}\, d^i$. A *torsionless* module is defined to be a submodule of a projective module.

Let us recall the definition of right/left/weakly Iwanaga–Gorenstein algebras.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. We say that an algebra Λ is *right (resp. left) Iwanaga–Gorenstein (abbr. RIG, resp. LIG)* if ${}^\perp\Lambda$ (resp. ${}^\perp\Lambda^{\mathrm{op}}$) is Frobenius, or equivalently, if any module in ${}^\perp\Lambda$ (resp. ${}^\perp\Lambda^{\mathrm{op}}$) is GP [RZ]. Here, Λ^{op} stands for the opposite algebra of Λ. A *weakly Iwanaga–Gorenstein (abbr. WIG)* algebra is defined to be both right and left IG. We call Λ *Iwanaga–Gorenstein (abbr. IG)* if it has finite left and right selfinjective dimension.

We give some examples of RIG algebras.

Example 3. [RZ] An algebra is RIG if it satisfies one of the following:

(1) it has finite left selfinjective dimension;

(2) there are only finitely many indecomposable torsionless modules in ${}^\perp\Lambda$.

In particular, an IG algebra is WIG.

Remark 4. [M1, RZ] We still do not know if RIG implies LIG, and vise versa. On the other hand, if RIG $\Leftrightarrow$ LIG, then the Gorenstein symmetry conjecture holds true; that is, $\mathrm{inj.dim}\,\Lambda_\Lambda < \infty \Leftrightarrow \mathrm{inj.dim}\,{}_\Lambda\Lambda < \infty$. To prove the statement, assume that $d := \mathrm{inj.dim}\,\Lambda_\Lambda < \infty$. By Example 3(2), we see that Λ is LIG, which implies that it is RIG by assumption. Let us show that $D\Lambda$ has finite projective dimension, where D stands for the K-dual. Denoting the syzygy of a module X by ΩX, we have an isomorphism $\mathrm{Ext}^i_\Lambda(\Omega^d D\Lambda, \Lambda) \simeq \mathrm{Ext}^{i+1}_\Lambda(\Omega^{d-1} D\Lambda, \Lambda) = 0$, which says that $X := \Omega^d D\Lambda$ belongs to ${}^\perp\Lambda$. Since Λ is RIG, it is seen that X is GP, whence it is given by $\Omega^{d+1}Y$ for some GP module Y up to projective summand. We now get isomorphisms

$$\begin{aligned}\mathrm{Ext}^1_\Lambda(\Omega^d Y, X) &\simeq \mathrm{Ext}^2_\Lambda(\Omega^{d-1}Y, X) \simeq \cdots \\ &\simeq \mathrm{Ext}^{d+1}_\Lambda(Y, X) \simeq \mathrm{Ext}^d_\Lambda(Y, \Omega^{d-1}D\Lambda) \simeq \cdots \simeq \mathrm{Ext}^1_\Lambda(Y, D\Lambda) = 0.\end{aligned}$$

Here, the isomorphisms in the second line follow from the condition of Y being GP. Therefore, it turns out that the exact sequence $0 \to X \to P \to \Omega^d Y \to 0$ with P projective splits, whence $X := \Omega^d D\Lambda$ is projective, and so $\mathrm{inj.dim}\,{}_\Lambda\Lambda \leq d < \infty$.

We denote by $\mathcal{C}_\Lambda$ the full subcategory of $\mathsf{mod}\,\Lambda$ consisting of torsionless modules which belong to ${}^\perp\Lambda$. Define

$$\Omega^\ell\,\mathcal{C}_\Lambda := \{X \in \mathsf{mod}\,\Lambda \mid X \simeq \Omega^\ell Y \text{ for some } Y \in \mathcal{C}_\Lambda\}.$$

A full subcategory of $\mathsf{mod}\,\Lambda$ is said to be *of finite type* if it has only finitely many indecomposable modules. Modifying Theorem 1.3 of [RZ] and its proof, we have the following.

Lemma 5. *If $\Omega^\ell\,\mathcal{C}_\Lambda$ is of finite type, then Λ is RIG and ${}^\perp\Lambda$ is also of finite type.*

Proof. Let X be in ${}^\perp\Lambda$. Note that ΩX belongs to $\mathcal{C}_\Lambda$. It follows that if X is indecomposable, then so is ΩX. By assumption, we see that $\Omega^r X$ is Ω-priodic for some $r \geq \ell$, which implies that it is GP. Hence, we deduce the fact that X is also GP, so Λ is RIG. □

We can now state the main theorem of this note.

Theorem 6. *Let B be an algebra and T a generator of B. Put $\Lambda := \mathrm{End}_\Lambda(T)$. If B is representation-finite, then Λ is WIG with ${}^\perp\Lambda$ of finite type.*

Marczinzik informed us that we can drop the assumption to be a generator [M2, Corollary 1.3(2)]; that is,

Theorem 7. *The endomorphism algebra of a module over a representation-finite algebra is WIG.*

Proof. Let M be a module over an algebra B and put $\Lambda := \mathrm{End}_B(M)$. It follows from [ARS, VI, Exercise 7] that $\Omega^2\,\mathcal{C}_\Lambda$ is of finite type, so Λ is RIG by Lemma 5. The dual argument leads to the fact that Λ is LIG, so WIG. □

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