ON THE GENDO-SYMMETRIC ALGEBRA OF A TRIVIAL EXTENSION ALGEBRA

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Abstract. We discuss the representation type of gendo-symmetric algebras. In particular, one gives a sufficient and necessary conditions that a gendo-symmetric algebra is representation-finite.

1. Introduction

Throughout this article, algebras are always assumed to be finite dimensional over an algebraically closed field K. By a module, we mean a finitely generated right module. For an algebra A, we denote by $\mathsf{mod}\,A$ the category of modules over A. The stable category of $\mathsf{mod}\,A$ is denoted by $\underline{\mathsf{mod}}\,A$. We denote the K-dual by $D := \mathrm{Hom}_K(-, K)$. We assume, unless otherwise stated, that all of our algebras are basic and connected.

In representation theory of algebras, endomorphism algebras play important roles. For example, the endomorphism algebra of a progenerator is Morita equivalent to the original algebra. More generally, the endomorphism algebra of a tilting module is derived equivalent to the original algebra. In this note, we consider the endomorphism algebra of a generator over a symmetric algebra, so-called a *gendo-symmetric algebra* [3]; note that gendo-symmetric algebra is not symmetric unless it is a progenerator.

We would like to classify the representation types of certain gendo-symmetric algebras. Note that, as the module category of an idempotent truncation embeds in the module category of the original algebra. If a gendo-symmetric algebra is representation-finite, then so is the original algebra. Let us recall the classification of representation-finite symmetric algebras.

Proposition 1. [1] *A representation-finite symmetric algebra is one of the following cases.*

- *A Brauer tree algebra;*
- *A modified Brauer tree algebra (See* [7]*);*
- *A representation-finite trivial extension algebra.*

The case of Brauer tree algebras was studied by Böhmler [2]. We will consider the case of trivial extension algebras.

This note is organized as follows. We treat the case of trivial extensions in Section 2 and its example in Section 3.

2. Trivial extension algebras

We start with recalling the definition of trivial extension algebras.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. The *trivial extension algebra* $T(A)$ of an algebra A is $A \oplus DA$ as a vector space. For $(a, f), (b, g) \in A \oplus DA = T(A)$, we define the multiplicity

$$(a, f) \cdot (b, g) := (ab, ag + fb).$$

We can completely describe when a trivial extensions is representation-finite.

Proposition 3. [4] *The trivial extension algebra $T(A)$ is representation-finite if and only if A is derived equivalent to a hereditary algebra of Dynkin type.*

The stable Auslander–Reiten (abbr. sAR) quiver of a representation-finite trivial extension algebra can be understood with translation quiver.

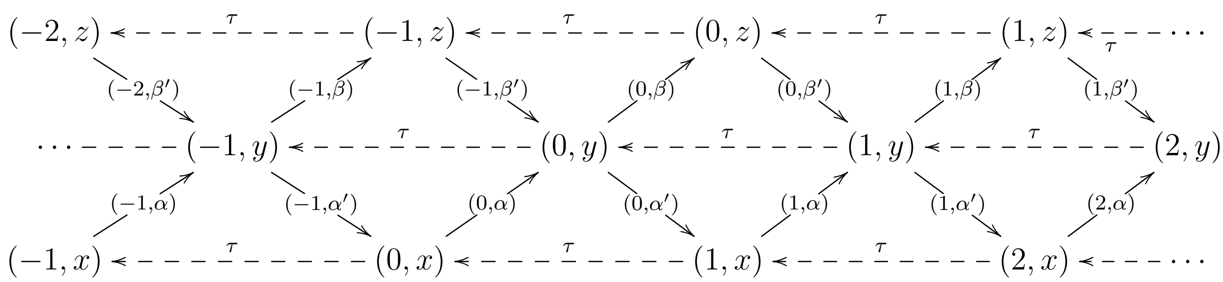
Definition 4. The *translation quiver* $\mathbb{Z}Q$ of a quiver $Q := (Q_0, Q_1, s, t)$ is defined as follows:

- $(\mathbb{Z}Q)_0 := \mathbb{Z} \times Q_0, (\mathbb{Z}Q)_1 := \mathbb{Z} \times Q_1 \cup \{(n, \alpha') \mid n \in \mathbb{Z}, \alpha \in Q_1\}$;
- For any arrow $(n, \alpha) \in \mathbb{Z} \times Q_1$,

$$\begin{aligned} (n, \alpha) : &\quad (n, s(\alpha)) \to (n, t(\alpha)), \\ (n, \alpha') : &\quad (n, t(\alpha)) \to (n + 1, s(\alpha)); \end{aligned}$$

- The translation $\tau : (\mathbb{Z}Q)_0 \to (\mathbb{Z}Q)_0$ is defined by $\tau((n, x)) = (n - 1, x)$.

Example 5. Let $Q := x \xrightarrow{\alpha} y \xrightarrow{\beta} z$. Then the translation quiver $\mathbb{Z}Q$ is as follows:



Definition 6. For a quiver Q and a group G generated by automorphisms of Q, the *orbit quiver* $Q/G := (Q_0/G, Q_1/G, \bar{s}, \bar{t})$ is defined by $\bar{s}(G(\alpha)) := G(s(\alpha)), \bar{t}(G(\alpha)) := G(t(\alpha))$.

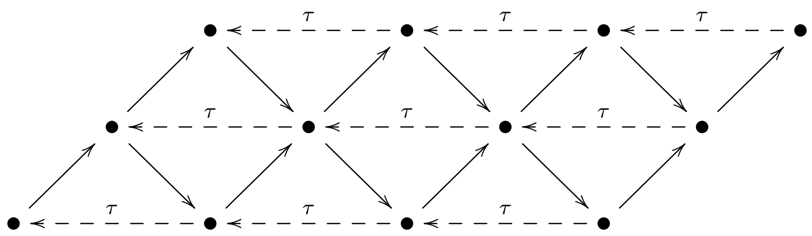
We describe of the sAR-quiver of a trivial extension algebra.

Proposition 7. [5, 6] *Let A be an algebra derived equivalent to a hereditary algebra of Dynkin type Δ. Then the sAR-quiver ${}_s\Gamma(\mathsf{mod}\, T(A))$ of $T(A)$ is isomorphic to $\mathbb{Z}\overrightarrow{\Delta}/\langle \tau^{h(\Delta)-1} \rangle$, where $h(\Delta)$ is the Coxeter number of Δ.*

Δ	$\mathbb{A}_n$	$\mathbb{D}_n$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$h(\Delta)$	$n+1$	$2(n-1)$	12	18	30

Remark 8. $\mathbb{Z}\overrightarrow{\Delta}$ does not depend on the orientation $\vec{\Delta}$ on Δ.

Example 9. Let $\overrightarrow{\Delta} := \overrightarrow{\mathbb{A}_3} = 1 \longrightarrow 2 \longrightarrow 3$. Then we have ${}_s\Gamma(\mathsf{mod}\, T(K\overrightarrow{\mathbb{A}_3})) \simeq \mathbb{Z}\overrightarrow{\mathbb{A}_3}/\langle\tau^3\rangle$, which takes the following form:



Here, we identify the left side and the right side.

In the rest of this section, we assume that A is derived equivalence to $K\overrightarrow{\Delta}$, where $\overrightarrow{\Delta}$ is a Dynkin quiver. To state the main theorem, we construct a quiver $\overrightarrow{\Delta}_M$ from $\overrightarrow{\Delta}$ and an indecomposable $T(A)$-module.

Definition 10. Let M be an indecomposable non-projective $T(A)$-module. Consider ${}_s\Gamma(\mathsf{mod}\, T(A)) \simeq \mathbb{Z}\overrightarrow{\Delta}/\langle\tau^{h(\Delta)-1}\rangle$, let us (x_M, y_M) be the vertex in $\mathbb{Z}\overrightarrow{\Delta}/\langle\tau^{h(\Delta)-1}\rangle$ corresponding to the position of M in the sAR-quiver. Then one defines a new quiver $\overrightarrow{\Delta}_M$ as follows:

- $(\overrightarrow{\Delta}_M)_0 := \{(x_M, i) \mid i \in \overrightarrow{\Delta}_0\} \cup \{\star\}$;
- $(\overrightarrow{\Delta}_M)_1 := \{(x_M, \alpha) \mid \alpha \in \overrightarrow{\Delta}_1\} \cup \{\beta : (x_M, y_M) \to \star\}$.

Now, we state our main theorem.

Theorem 11. *Let M be as in Definition 10. Then the gendo-symmetric algebra $\Lambda := \mathrm{End}_{T(A)}(T(A) \oplus M)$ is isomorphic to $T(B)/\operatorname{soc} P_\star$, where B is derived equivalent to $K\overrightarrow{\Delta}_M$ and $P_\star$ is an indecomposable projective $T(B)$-module corresponding to the new vertex $\star$. In particular, Λ is representation-finite if and only if $\overrightarrow{\Delta}_M$ is a Dynkin quiver.*

3. Examples

We display a module by writing down its Loewy series from top to bottom.

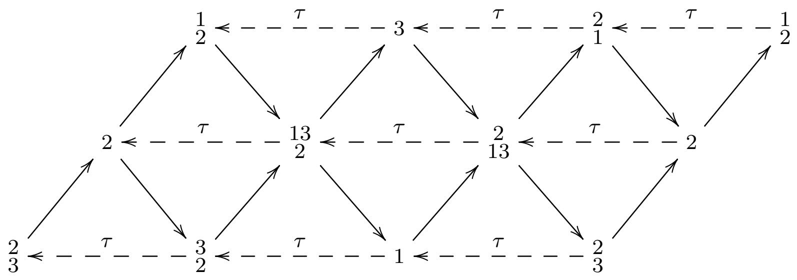
Example 12. Let A be an algebra with the following quiver and relations:

$$(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3),\ \alpha\beta = 0.$$

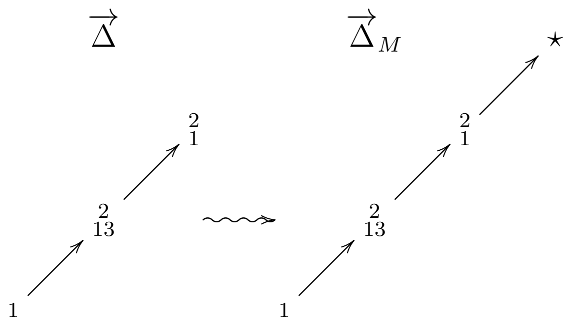
The trivial extension algebra $T(A)$ is given by the following quiver and relations:

$$(1 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3),\quad \begin{array}{c} \alpha\beta = \beta'\alpha' = 0 \\ \alpha'\alpha = \beta\beta'. \end{array}$$

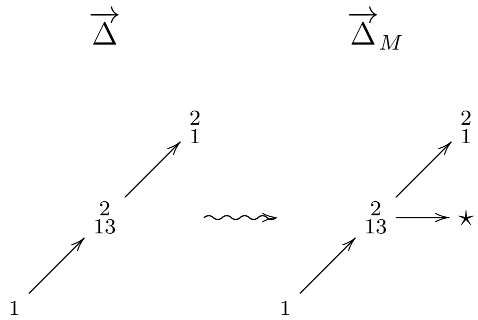
Since A is derived equivalent to $K\overrightarrow{\mathbb{A}_3}$, the sAR-quiver of $T(A)$ is isomorphic to $\mathbb{Z}\overrightarrow{\mathbb{A}}_3/\langle\tau^3\rangle$, so it has the form:



(1) Put $M = {}^{2}_{1}$. By Theorem 11, the gendo-symmetric algebra $\Lambda := \mathrm{End}_{T(A)}(T(A) \oplus M)$ is representation-finite, where $\overrightarrow{\Delta}_M = \overrightarrow{\mathbb{A}_4}$ is constructed as follows:

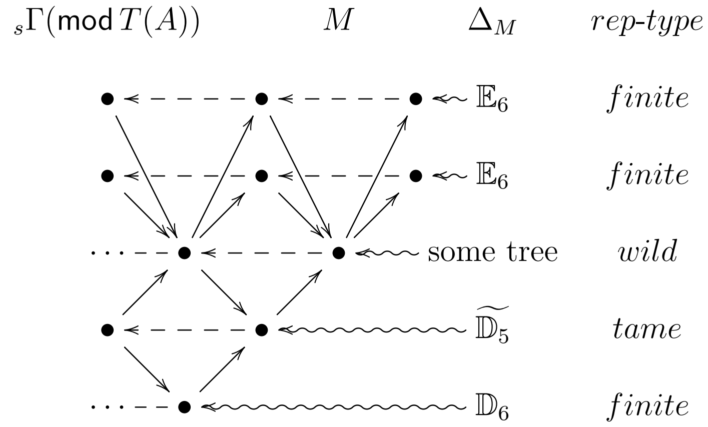


(2) Put $N = {}^{2}_{13}$. By Theorem 11, the gendo-symmetric algebra $\Gamma := \mathrm{End}_{T(A)}(T(A) \oplus N)$ is representation-finite, where $\overrightarrow{\Delta}_M = \overrightarrow{\mathbb{D}_4}$ is constructed as follows:



Moreover for any indecomposable module L, the gendo-symmetric algebra of L over $T(A)$ is representation-finite; because $\overrightarrow{\Delta}_M$ is $\overrightarrow{\mathbb{A}_4}$ or $\overrightarrow{\mathbb{D}_4}$.

Example 13. Let A be an algebra derived equivalent to $K\overrightarrow{\mathbb{D}_5}$. For each module M, we give the representation type of the gendo-symmetric algebra $\Lambda := \mathrm{End}_{T(A)}(T(A) \oplus M)$:



Here, the arrow $\rightsquigarrow$ points out the position of M, and we then have the quiver $\overrightarrow{\Delta_M}$ of Λ. Consequently, we get the representation-type of Λ.

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