WIDE SUBCATEGORIES AND LATTICES OF TORSION CLASSES

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Abstract. The partially ordered set $\operatorname{tors}\mathcal{A}$ of torsion classes in a fixed abelian length category $\mathcal{A}$ is a complete lattice. For two torsion classes $\mathcal{U} \subset \mathcal{T}$, the interval $[\mathcal{U}, \mathcal{T}]$ in $\operatorname{tors}\mathcal{A}$ is a sublattice of $\operatorname{tors}\mathcal{A}$, and the subcategory $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$ describes the "width" of the interval $[\mathcal{U}, \mathcal{T}]$. Motivated by τ-tilting reduction of Jasso, we mainly deal with the case that $\mathcal{W}$ is a wide subcategory of $\mathcal{A}$; we call such intervals wide intervals. Our first main result in this proceeding claims that a wide interval $[\mathcal{U}, \mathcal{T}]$ is isomorphic to $\operatorname{tors}\mathcal{W}$ of torsion classes in the abelian category $\mathcal{W}$. Moreover, we give some characterizations of wide intervals in terms of the Hasse quiver of the lattice $\operatorname{tors}\mathcal{A}$. This proceeding is based on the joint work [3] with Calvin Pfeifer (Universität Bonn).

1. Preliminary

Throughout this proceeding, we assume that $\mathcal{A}$ is an (essentially small) abelian length category. Therefore, any object $X \in \mathcal{A}$ has a composition series $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ with each X_i/X_{i-1} ($i \in \{1, 2, \ldots, n\}$) is a simple object in $\mathcal{A}$. All subcategories in this proceeding are supposed to be full subcategories.

We first recall the definition of torsion pairs by Dickson.

Definition 1. [6] Let $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$ be full subcategories. Then, the pair $(\mathcal{T}, \mathcal{F})$ is called a *torsion pair* in $\mathcal{A}$ if

$$\mathcal{F} = \mathcal{T}^{\perp} := \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, X) = 0\},$$
$$\mathcal{T} = {}^{\perp}\mathcal{F} := \{X \in \mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(X, \mathcal{F}) = 0\}.$$

Torsion pairs can be characterized in terms of short exact sequences as follows.

Lemma 2. *Let $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$ be full subcategories. Then, the pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$ if and only if $\operatorname{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$ and every $X \in \mathcal{A}$ admits a short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.*

In this proceeding, we mainly focus on subcategories $\mathcal{T}$ which can be completed to a torsion pair $(\mathcal{T}, \mathcal{F})$.

Definition 3. A full subcategory $\mathcal{T} \subset \mathcal{A}$ is called a *torsion class* in $\mathcal{A}$ if there exists a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$. We write $\operatorname{tors}\mathcal{A}$ for the set of torsion classes in $\mathcal{A}$.

We regard the set $\operatorname{tors}\mathcal{A} = (\operatorname{tors}\mathcal{A}, \subset)$ of torsion classes as a partially ordered set by inclusion. We give some fundamental observations for this proceeding.

Lemma 4. *We have the following properties.*

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(1) *Let $\mathcal{T} \subset \mathcal{A}$ be a full subcategory. Then, $\mathcal{T}$ is a torsion class if and only if $\mathcal{T}$ is closed under factor objects and extensions.*
(2) *For any $\mathcal{X} \subset \mathcal{A}$, there exists a smallest torsion class containing $\mathcal{X}$, which is denoted by $\mathsf{T}(\mathcal{X})$.*
(3) *The partially ordered set $\mathsf{tors}\,\mathcal{A}$ is a complete lattice with the join and the meet for each $S \subset \mathsf{tors}\,\mathcal{A}$ is given by*

$$\bigvee_{\mathcal{T} \in S} \mathcal{T} = \mathsf{T}\left(\bigcup_{\mathcal{T} \in S} \mathcal{T}\right), \quad \bigwedge_{\mathcal{T} \in S} \mathcal{T} = \bigcap_{\mathcal{T} \in S} \mathcal{T}.$$

2. Wide intervals

For two torsion classes $\mathcal{U} \subset \mathcal{T}$, we can naturally consider the interval

$$[\mathcal{U}, \mathcal{T}] := \{\mathcal{V} \in \mathsf{tors}\,\mathcal{A} \mid \mathcal{U} \subset \mathcal{V} \subset \mathcal{T}\}.$$

in $\mathsf{tors}\,\mathcal{A}$. The "width" of this interval is described by the full subcategory $\mathcal{U}^{\perp} \cap \mathcal{T} \subset \mathcal{A}$.

In this proceeding, we mainly deal with the following nice intervals.

Definition 5. [3, Definition 4.1] An interval $[\mathcal{U}, \mathcal{T}]$ in $\mathsf{tors}\,\mathcal{A}$ is called a *wide interval* if the full subcategory $\mathcal{U}^{\perp} \cap \mathcal{T}$ is a wide subcategory of $\mathcal{A}$.

Here, we say that a full subcategory $\mathcal{W} \subset \mathcal{A}$ is *wide* if $\mathcal{W}$ is closed under taking factor objects, subobjects, and extensions; or equivalently, $\mathcal{W}$ is an abelian subcategory of $\mathcal{A}$ closed under extensions. In particular, we can consider the complete lattice $\mathsf{tors}\,\mathcal{W}$ of torsion classes in the abelian category $\mathcal{W}$. The set of isoclasses of simple objects of a wide subcategory $\mathcal{W}$ is a *semibrick*, that is, a set of pairwise Hom-orthogonal isoclasses of bricks. Conversely, for each semibrick $\mathcal{S}$, the filtration closure $\mathsf{Filt}\,\mathcal{S}$ in $\mathcal{A}$ is a wide subcategory of $\mathcal{A}$. The wide subcategories of $\mathcal{A}$ bijectively correspond to the semibricks in $\mathcal{A}$ in this way [10, 1.2].

In particular, for any $\mathcal{T} \in \mathsf{tors}\,\mathcal{A}$, $[\mathcal{T}, \mathcal{T}]$ is a wide interval, since $\mathcal{T}^{\perp} \cap \mathcal{T} = \{0\}$ is a wide subcategory, whose corresponding semibrick is the emptyset.

Later in this section, we will state our reduction theorem of wide intervals, which is an extension of results on τ-tilting reduction by Jasso [8] and Demonet–Iyama–Reading–Reiten–Thomas [5] to arbitrary wide intervals.

We recall that two torsion classes $\mathcal{U} \subset \mathcal{T} \in \mathsf{tors}\,\mathcal{A}$ are said to be *adjacent* if $\mathcal{U} \neq \mathcal{T}$ and there exists no torsion class $\mathcal{V} \in \mathsf{tors}\,\mathcal{A}$ such that $\mathcal{U} \subsetneq \mathcal{V} \subsetneq \mathcal{T}$. The adjacency relations of torsion classes in $\mathcal{A}$ is expressed by the *Hasse quiver* of the partially ordered set $\mathsf{tors}\,\mathcal{A}$, which is the quiver whose vertices are the elements of $\mathsf{tors}\,\mathcal{A}$ and there exists an arrow $\mathcal{T} \to \mathcal{U}$ if and only if $\mathcal{U} \subset \mathcal{T}$ are adjacent.

The following property is very crucial to introduce *brick labeling* in the sense of Demonet–Iyama–Reading–Reiten–Thomas [5]. This says that adjacent torsion classes give a minimal nontrivial wide interval.

Proposition 6. [5, Theorem 3.3] *For any arrow $q\colon \mathcal{T} \to \mathcal{U}$, the interval $[\mathcal{U}, \mathcal{T}]$ is a wide interval, and the associated wide subcategory $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$ has only one brick S_q up to isomorphisms. Thus, we label the arrow $q\colon \mathcal{T} \to \mathcal{U}$ by the brick S_q.*

We remark that, for any interval $[\mathcal{U},\mathcal{T}]$ in $\mathsf{tors}\,\mathcal{A}$, we can define the Hasse quiver of $[\mathcal{U},\mathcal{T}]$ in the same way as before. Then, the Hasse quiver of $[\mathcal{U},\mathcal{T}]$ is a full subquiver of the Hasse quiver of $\mathsf{tors}\,\mathcal{A}$, since the interval $[\mathcal{U},\mathcal{T}]$ is a convex subset of $\mathsf{tors}\,\mathcal{A}$.

To give another example of wide intervals, we recall some notions on τ-tilting theory for finite-dimensional algebras introduced in [1].

Let A be a finite-dimensional algebra over a field K, and set $\mathcal{A}$ as the category $\mathsf{mod}\,A$ of finite-dimensional A-modules. For $N, Q \in \mathsf{mod}\,A$ with Q projective, the pair (N, Q) is a *τ-rigid pair* if $\mathsf{Hom}_A(N, \tau N) = 0$ and $\mathsf{Hom}_A(Q, N) = 0$. Here, τ denotes the Auslander–Reiten translation in $\mathsf{mod}\,A$.

Then, by the following theory by Jasso [8] and Demonet–Iyama–Reading–Reiten–Thomas [5] called *τ-tilting reduction*, we can construct a wide interval for each τ-rigid pair.

Theorem 7. [8, Theorems 3.8, 3.12] [5, Theorem 4.12, Proposition 4.13] *For a τ-rigid pair (N, Q) in $\mathsf{mod}\,A$, set two torsion classes $\mathcal{U} \subset \mathcal{T}$ by*

$$\mathcal{U} := \mathsf{Fac}\,N, \quad \mathcal{T} := N^{\perp} \cap {}^{\perp}(\tau N) \cap Q^{\perp}.$$

Then, the following assertions hold.

(1) *The interval $[\mathcal{U},\mathcal{T}]$ is a wide interval.*

(2) *Set $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$. Then, $[\mathcal{U},\mathcal{T}]$ is isomorphic to $\mathsf{tors}\,\mathcal{W}$ as complete lattices by*

$$\Phi\colon [\mathcal{U},\mathcal{T}] \to \mathsf{tors}\,\mathcal{W}, \quad \mathcal{V} \mapsto \mathcal{U}^{\perp} \cap \mathcal{V}.$$

The inverse isomorphism is given by $\mathsf{tors}\,\mathcal{W} \ni \mathcal{X} \mapsto \mathsf{T}(\mathcal{U} \cup \mathcal{X}) \in [\mathcal{U},\mathcal{T}]$. Therefore, the Hasse quivers of $[\mathcal{U},\mathcal{T}]$ and $\mathsf{tors}\,\mathcal{W}$ are isomorphic.

(3) *The isomorphisms in (2) preserve brick labeling of the Hasse quivers; that is, the label of each arrow $\mathcal{V}_1 \to \mathcal{V}_2$ in $[\mathcal{U},\mathcal{T}]$ is the same as the label of the arrow $\Phi(\mathcal{V}_1) \to \Phi(\mathcal{V}_2)$ in $\mathsf{tors}\,\mathcal{W}$.*

Moreover, they showed that there exists a finite-dimensional K-algebra C such that $\mathcal{W} \cong \mathsf{mod}\,C$, which can be constructed from the Bongartz completion of the τ-rigid pair (N, Q).

Now, we can state our first main result, which says that the parts (2) and (3) in the previous theorem actually hold for all wide intervals.

Theorem 8. [3, Theorem 4.2] *Let $[\mathcal{U},\mathcal{T}]$ is a wide interval in $\mathsf{tors}\,\mathcal{A}$ and set $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$. Then, the following assertions hold.*

(1) *The interval $[\mathcal{U},\mathcal{T}]$ is isomorphic to $\mathsf{tors}\,\mathcal{W}$ as complete lattices by*

$$\Phi\colon [\mathcal{U},\mathcal{T}] \to \mathsf{tors}\,\mathcal{W}, \quad \mathcal{V} \mapsto \mathcal{U}^{\perp} \cap \mathcal{V}.$$

The inverse isomorphism is given by $\mathsf{tors}\,\mathcal{W} \ni \mathcal{X} \mapsto \mathsf{T}(\mathcal{U} \cup \mathcal{X}) \in [\mathcal{U},\mathcal{T}]$. Therefore, the Hasse quivers of $[\mathcal{U},\mathcal{T}]$ and $\mathsf{tors}\,\mathcal{W}$ are isomorphic.

(2) *The isomorphisms in (1) preserve brick labeling of the Hasse quivers; that is, the label of each arrow $\mathcal{V}_1 \to \mathcal{V}_2$ in $[\mathcal{U},\mathcal{T}]$ is the same as the label of the arrow $\Phi(\mathcal{V}_1) \to \Phi(\mathcal{V}_2)$ in $\mathsf{tors}\,\mathcal{W}$.*

(3) *The following three sets coincide:*
   (a) *the set of labels of the arrows from $\mathcal{T}$ in $[\mathcal{U},\mathcal{T}]$;*
   (b) *the set of labels of the arrows to $\mathcal{U}$ in $[\mathcal{U},\mathcal{T}]$;*
   (c) *the set of isoclasses of the simple objects of $\mathcal{W}$.*

In the following example, we give a wide interval which does not come from τ-tilting reduction.

Example 9. [3, Example 4.3] Let A be the Kronecker quiver algebra $K(1 \rightrightarrows 2)$ over an algebraically closed field K, and set $\mathcal{A} := \mathsf{mod}\, A$. We set two torsion classes $\mathcal{U}, \mathcal{T} \subset \mathcal{A}$ so that

- $\mathcal{U}$ is the smallest torsion class containing all the preinjective modules in $\mathsf{mod}\, A$; and that
- $\mathcal{T}$ is the smallest torsion class containing all the regular modules and all the preinjective modules in $\mathsf{mod}\, A$.

Then, $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$ is a wide subcategory of $\mathsf{mod}\, A$, and its simple objects are all the quasi-simple regular modules; namely,

$$M_\lambda := \ K \underset{b}{\overset{a}{\rightrightarrows}} K \quad (\lambda = (a : b) \in \mathbb{P}^1(K)).$$

Thus, $[\mathcal{U}, \mathcal{T}]$ is a wide interval. Since $\mathsf{Ext}^1_A(M_\lambda, M_\mu) = 0$ if $\lambda \neq \mu$, we get

$$\mathcal{W} \cong \bigoplus_{\lambda \in \mathbb{P}^1(K)} \mathsf{Filt}\, M_\lambda.$$

It is easy to see that $\mathsf{tors}(\mathsf{Filt}\, M_\lambda) = \{\mathsf{Filt}\, M_\lambda, \{0\}\}$. Therefore, from Theorem 8, we have

$$[\mathcal{U}, \mathcal{T}] \cong \mathsf{tors}\, \mathcal{W} \cong \prod_{\lambda \in \mathbb{P}^1(K)} \mathsf{tors}(\mathsf{Filt}\, M_\lambda) \cong 2^{\mathbb{P}^1(K)}$$

as lattices, where the corresponding element in $[\mathcal{U}, \mathcal{T}]$ to each $X \in 2^{\mathbb{P}^1(K)}$ is

$$\mathcal{V}_X := \mathsf{T}(\mathcal{U} \cup \{M_\lambda \mid \lambda \in X\}) \in [\mathcal{U}, \mathcal{T}].$$

Any arrow in the Hasse quiver of $[\mathcal{U}, \mathcal{T}]$ is of the form

$$\mathcal{V}_{X \cup \{\lambda\}} \xrightarrow{\text{label: } M_\lambda} \mathcal{V}_X \quad (X \in 2^{\mathbb{P}^1(K)},\ \lambda \in \mathbb{P}^1(K) \setminus X).$$

3. Characterizations of wide intervals

Next, we will characterize wide intervals in a combinatorial way. For this purpose, we define the following notions.

Definition 10. [3, Definition 5.1] Let $[\mathcal{U}, \mathcal{T}]$ be an interval in $\mathsf{tors}\, \mathcal{A}$.

(1) We set

$$[\mathcal{U}, \mathcal{T}]^- := \{\mathcal{U}\} \cup \{\mathcal{V} \in [\mathcal{U}, \mathcal{T}] \mid \text{there exists an arrow } \mathcal{V} \to \mathcal{U}\},$$
$$[\mathcal{U}, \mathcal{T}]^+ := \{\mathcal{T}\} \cup \{\mathcal{V} \in [\mathcal{U}, \mathcal{T}] \mid \text{there exists an arrow } \mathcal{T} \to \mathcal{V}\}.$$

(2) The interval $[\mathcal{U}, \mathcal{T}]$ is called a *join interval* if

$$\mathcal{T} = \bigvee_{\mathcal{V} \in [\mathcal{U}, \mathcal{T}]^-} \mathcal{V}.$$

(3) The interval $[\mathcal{U},\mathcal{T}]$ is called a *meet interval* if

$$\mathcal{U} = \bigwedge_{\mathcal{V}\in[\mathcal{U},\mathcal{T}]^+} \mathcal{V}.$$

Note that join intervals and meet intervals are purely lattice theoritical notions. We showed that actually they coincide with wide intervals.

Theorem 11. [3, Theorem 5.2] *Let $[\mathcal{U},\mathcal{T}]$ be an interval in $\mathsf{tors}\,\mathcal{A}$. Then, the following conditions are equivalent:*

(a) *$[\mathcal{U},\mathcal{T}]$ is a wide interval;*
(b) *$[\mathcal{U},\mathcal{T}]$ is a join interval;*
(c) *$[\mathcal{U},\mathcal{T}]$ is a meet interval.*

Next, we consider the following question:

> Fix $\mathcal{T} \in \mathsf{tors}\,\mathcal{A}$, then how many torsion classes $\mathcal{U} \in \mathsf{tors}\,\mathcal{A}$ satisfy that $[\mathcal{U},\mathcal{T}]$ are wide intervals?

To answer this, it is useful to use the subcategory

$$\alpha(\mathcal{T}) := \{X \in \mathcal{T} \mid \text{for all } Y \in \mathcal{T} \text{ and all } f\colon Y \to X,\ \mathsf{Ker}\,f \in \mathcal{T}\}$$

associated to each $\mathcal{T} \in \mathsf{tors}\,\mathcal{A}$. Ingalls–Thomas [7, Proposition 2.12] showed that $\alpha(\mathcal{T})$ is a wide subcategory, and they used this wide subcategory efficiently to study the relationship between wide subcategories and torsion classes. In the case $\mathcal{A} = \mathsf{mod}\,A$ with A a finite-dimensional hereditary algebra, [7, Proposition 2.14] showed that $\alpha(\mathsf{T}(\mathcal{W})) = \mathcal{W}$ for any wide subcategory $\mathcal{W} \subset \mathcal{A}$, and [9, Proposition 3.3] extended this to the case that A is an arbitrary finite-dimensional K-algebra. We remark that the proof of [9] works also in our setting.

By using the operation α, we have obtained the following properties on the number of wide intervals.

Theorem 12. [3, Proposition 6.5, Theorem 6.7] *Fix $\mathcal{T} \in \mathsf{tors}\,\mathcal{A}$, and set $\mathcal{L}$ as the set of labels of the arrows from $\mathcal{T}$ in the Hasse quiver of $\mathsf{tors}\,\mathcal{A}$. Then, the following assertions hold.*

(1) *The set $\mathcal{L}$ is a semibrick with $\mathsf{Filt}\,\mathcal{L} = \alpha(\mathcal{T})$.*
(2) *There exists a bijection*

$$2^{\mathcal{L}} \to \{\mathcal{U} \in \mathsf{tors}\,\mathcal{A} \mid [\mathcal{U},\mathcal{T}] \text{ is a wide interval}\},$$
$$\mathcal{S} \mapsto \mathcal{T} \cap {}^{\perp}\mathcal{S} =: \mathcal{U}_{\mathcal{S}}.$$

Moreover, $(\mathcal{U}_{\mathcal{S}})^{\perp} \cap \mathcal{T} = \mathsf{Filt}\,\mathcal{S}$ holds for any $\mathcal{S} \in 2^{\mathcal{L}}$, and it is a Serre subcategory of $\alpha(\mathcal{T})$.

As an application of the theorem above, we found the following criterion, which determines whether a given torsion class $\mathcal{T} \in \mathsf{tors}\,\mathcal{A}$ admits a wide subcategory $\mathcal{W} \subset \mathcal{A}$ such that $\mathcal{T} = \mathsf{T}(\mathcal{W})$. We call such torsion classes *widely generated torsion classes.*

Corollary 13. [3, Theorem 7.2] *For $\mathcal{T} \in \mathsf{tors}\,\mathcal{A}$, set $\mathcal{L}$ as the set of labels of the arrows from $\mathcal{T}$. Then, the following conditions are equivalent:*

(a) $\mathcal{T}$ *is a widely generated torsion class;*
(b) $\mathcal{T} = \mathsf{T}(\alpha(\mathcal{T}))$*;*
(c) $\mathcal{T}$ *coincides with* $\mathsf{T}(\mathcal{L})$*;*
(d) *for any torsion class* $\mathcal{U} \in \mathsf{tors}\,\mathcal{A}$ *satisfying* $\mathcal{U} \subset \mathcal{T}$*, there exists an arrow* $\mathcal{T} \to \mathcal{U}'$ *such that* $\mathcal{U} \subset \mathcal{U}'$*.*

We remark that the equivalence of the conditions (a), (c), and (d) above has been already proved by Barnard–Carroll–Zhu [4, Subsection 3.2] using *minimal extending modules*.

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