COHEN-MACAULAY MODULES OVER YONEDA ALGEBRAS

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ABSTRACT. For a finite dimensional algebra Λ of finite representation type and an additive generator M for $\operatorname{mod}\Lambda$, we investigate ring theoretic properties and representation theory of the Yoneda algebra $\Gamma = \bigoplus_{i\geq 0}\operatorname{Ext}^i_\Lambda(M,M)$. We give fundamental results on these Γ, such as coherence, Gorenstein property, and a description of the stable category of Cohen-Macaulay modules.

1. Setting

Yoneda algebras form a class of algebras which has long been studied in ring theory and representation theory. They are defined for a ring Λ and a Λ-module M by the formula

$$\Gamma = \bigoplus_{i\geq 0}\operatorname{Ext}^i_\Lambda(M,M),$$

with multiplication given by the Yoneda product. Such algebras have been of great interest typically in the context of Koszul duality, Hochschild cohomology, and so on. The aim of this note is to study the Yoneda algebra Γ in the following setup, which is different from above and serves as the main definition of this article.

Setting 1. (1) Λ is a finite dimensional algebra of finite representation type.
(2) M is the additive generator for $\operatorname{mod}\Lambda$.

We call Γ the *Yoneda algebra of* Λ since it is determined by Λ up to graded Morita equivalence. Note that our Γ is finite dimensional if and only if Λ has finite global dimension, and in general Γ is even far from being Noetherian.

2. Results

Throughout, let Λ and M be as in Setting 1, and Γ the Yoneda algebra of Λ. Let us recall some definitions needed to state our result.

Definition 2. (1) A graded ring R is called *graded coherent* if the category $\operatorname{mod}^{\mathbb{Z}}R$ (resp. $\operatorname{mod}^{\mathbb{Z}}R^{\mathrm{op}}$) of finitely presented left (resp. right) modules is abelian.
(2) A graded coherent ring R is *d-Gorenstein* if $\operatorname{inj.dim}R \leq d$ in $\operatorname{mod}^{\mathbb{Z}}R^{\mathrm{op}}$ and in $\operatorname{mod}^{\mathbb{Z}}R^{\mathrm{op}}$.
(3) Let R be a graded coherent Gorenstein ring. The category $\mathrm{CM}^{\mathbb{Z}}R$ of graded *Cohen-Macaulay modules* is defined by

$$\mathrm{CM}^{\mathbb{Z}}R = \{X \in \operatorname{mod}^{\mathbb{Z}}R \mid \operatorname{Ext}^i_R(X,R)=0 \text{ for all } i>0\}.$$

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The definitions (2)(3) are just adaptions of the usual ones for Noetherian case to coherent setting. As usual, the category $\mathrm{CM}^{\mathbb{Z}}R$ is naturally a Frobenius category and hence the stable category $\underline{\mathrm{CM}}^{\mathbb{Z}}R$ is triangulated. Moreover it is canonically equivalent to the *singularity category* $\mathrm{D^b}(\mathrm{mod}^{\mathbb{Z}}R)/\mathrm{K^b}(\mathrm{proj}^{\mathbb{Z}}R)$ which is studied also in algebraic geometry as an important invariant of singularities.

One of the most fascinating problems concerning Cohen-Macaulay representation theory over Gorenstein rings is to find equivalences between triangulated categories, and recently there have been extensive studies and various results have been established, see [3] and references therein.

Our main results state that we can and do place ourselves into the above context for the Yoneda algebras.

Theorem 3. *Let Λ be an arbitrary finite dimensional algebra of finite representation type and Γ the Yoneda algebra of Λ.*

(1) *Γ is graded coherent.*
(2) *Γ is 1-Gorenstein.*
(3) *Letting $\underline{\Gamma}$ be the stable Auslander algebra $\underline{\mathrm{End}}_\Lambda(M)$ of Λ, there is a triangle equivalence*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}\Gamma \simeq \mathrm{D^b}(\mathrm{mod}\,\underline{\Gamma}).$$

Let us given a brief account of our discussion. The starting point is to consider the subcategory

$$\mathcal{Y} = \mathrm{add}\{M[i] \mid i \in \mathbb{Z}\} \subset \mathrm{D^b}(\mathrm{mod}\,\Lambda).$$

Since we can rewrite $\Gamma = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{D^b}(\Lambda)}(M, M[i])$, there is an equivalence $\mathcal{Y} \simeq \mathrm{proj}^{\mathbb{Z}}\Gamma$, thus we can regard $\mathcal{Y}$ as the categorical analogue of the Yoneda algebra Γ. Note also that this description shows that the assumption 'Λ is representation-finite' is not essential, and all the results can be generalized to arbitrary finite dimensional algebras in terms of the 'category $\mathcal{Y}$' instead of the 'graded algebra Γ'.

Applying this description, one obtains the coherence of Γ by the following (non-trivial) observation.

Proposition 4. *The subcategory $\mathcal{Y}$ is functorially finite in* $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$.

Now we turn to the equivalence (3). Note first that in general the global dimension of the stable Auslander algebra $\underline{\Gamma}$ is infinite, hence there is *no* tilting object in $\underline{\mathrm{CM}}^{\mathbb{Z}}\Gamma$. Therefore we need another strategy than finding a tilting object to build an equivalence. We use the technique of *realization functors* [1, 5] in the following sense.

Definition 5. Let $\mathcal{T}$ be a triangulated category with a t-structure whose heart is $\mathcal{H}$. A *realization functor* is a triangle functor $\mathrm{D^b}(\mathcal{H}) \to \mathcal{T}$ extending the inclusion $\mathcal{H} \subset \mathcal{T}$.

An immediate question on the existence of a realization functor is settled for a reasonably large class of triangulated categories.

Fact 6 ([4] etc)**.** *Realization functors exist for algebraic triangulated categories.*

Thanks to this fact, we can use a realization functor on our algebraic triangulated category $\underline{\mathrm{CM}}^{\mathbb{Z}}\Gamma$. We refer to [2] for detailed discussions.

We end this article with an example.

Example 7. Let $\Lambda = k[x]/(x^{n+1})$. This is a representation-finite self-injective algebra whose AR-quiver is

$$1 \rightleftarrows 2 \rightleftarrows \cdots \rightleftarrows n-1 \rightleftarrows n \rightleftarrows n+1\ ,$$

where each vertex is labeled by the length of the corresponding indecomposable module. One can verify that the Yoneda algebra Γ of Λ is presented by the following graded quiver with relations:

$$\begin{tikzcd}
1 \arrow[r, shift left, "a"] \arrow[rrrr, bend left=30, "x"] & 2 \arrow[l, shift left, "b"] \arrow[r, shift left, "a"] \arrow[rr, bend left, "x"] & \cdots \arrow[l, shift left, "b"] \arrow[r, shift left, "a"] & l-1 \arrow[l, shift left, "b"] \arrow[r, shift left, "a"] \arrow[ll, bend left, "x"] & l \arrow[l, shift left, "b"] \arrow[r, shift left, "p"] \arrow[llll, bend left=30, "x"] & l+1 \arrow[l, shift left, "q"]
\end{tikzcd},$$

$$\deg a = \deg b = \deg p = \deg q = 0,\ \deg x = 1,$$
$$ab = ba,\ ba = pq, \quad ax = xb,\ bx = xa, \quad xp = 0,\ qx = 0.$$

Let us apply Theorem 3 to this algebra. By (1) we see that Γ is graded coherent, but this is in fact Noetherian (since Λ is self-injective). Together with (2) we deduce that Γ is 1-Gorenstein. Now we turn to (3). We have that the stable Auslander algebra of Λ is the preprojective algebra Π of type A_n. Therefore we obtain a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}}\Gamma \simeq \mathrm{D^b}(\mathrm{mod}\,\Pi)$.

References

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