ON THE 2-TEST MODULES FOR PROJECTIVITY AND WEAKLY $\mathfrak{m}$-FULL IDEALS

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Abstract. In this paper, we introduce the notion of an n-test module for projectivity and an n-Tor-test module for projectivity, and we show that the weakly $\mathfrak{m}$-full ideal with a few assumption is 2-test module for projectivity.

1. Introduction

Throughout this paper, let R be a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field k. All modules considered in this paper are assumed to be finitely generated. The notion of a strong test module for projectivity has been introduced and studied by Ramras [7]. An R-module M is called a *strong test module for projectivity* if every R-module N with $\mathrm{Ext}^1_R(N, M) = 0$ is projective. The residue field k is a typical example of a strong test module for projectivity. Ramras shows that the maximal ideal $\mathfrak{m}$ is a strong test module for projectivity. He also proves that every strong test module for projectivity has depth at most one. Jothilingam [6] proves that when R is a regular local ring, every R-module of depth at most one is a strong test module for projectivity. Araya, Iima and Takahashi [1] yields that the converse of this Jothilingam's result also holds true.

First of all, let us make the following definition.

Definition 1. Let M be a non-zero module and let n be a positive integer.

(1) M is called *n-test module for projectivity* if every module X with $\mathrm{Ext}^{1\sim n}_R(X, M) = 0$ is projective.
(2) M is called *n-Tor-test module for projectivity* if every module X with $\mathrm{Tor}^R_{1\sim n}(X, M) = 0$ is projective.
(3) An ideal I is called *weakly $\mathfrak{m}$-full* if I equals to the ideal $\mathfrak{m}I : \mathfrak{m}$.

Here let me give typical examples.

Example 2. Let n be a positive integer.

(1) k and $\mathfrak{m}$ are 1-test modules for projectivity, therefore n-test modules for projectivity for any positive integer n.
(2) k is a 1-Tor-test module for projectivity, therefore n-Tor-test modules for projectivity for any positive integer n.
(3) For an ideal I, if either I is integrally closed or R/I has positive depth, then I is weakly $\mathfrak{m}$-full.

Let me show several results.

The detailed version of this paper will be submitted for publication elsewhere.

Proposition 3. *Let $x \in \mathfrak{m}$ be a non-zero-divisor over an R-module M.*

(1) *If M is an n-test module for projectivity then M/xM is an n-test R-module for projectivity.*
(2) *If M/xM is an n-test R-module for projectivity then M is an $(n+1)$-test module for projectivity.*
(3) *If M is an n-Tor-test module for projectivity then M/xM is an n-Tor-test R-module for projectivity.*
(4) *If M is an n-test module for projectivity then the syzygy module of $M(=: \Omega_R M)$ is an $(n+2)$-test module for projectivity.*
(5) *If R is an integrally closed domain and $\Omega_R M$ is an n-test module for projectivity then M is an $(n+2)$-test module for projectivity.*
(6) *Every n-test module for projectivity has depth at most n.*

The main results in this paper are the following three theorems.

Theorem 4. *If M is an n-Tor-test module for projectivity then $M, \Omega_R M, \Omega_R^2 M, \ldots, \Omega_R^n M$ are n-test modules for projectivity.*

Theorem 5. *If I is weakly $\mathfrak{m}$-full and $\mathrm{Tor}_1^R(M, R/I) = 0$ then a free covering $0 \to N \to F \to M \to 0$ induces a short exact sequence $0 \to N/IN \to F/IF \to M/IM \to 0$ satisfying $\mathrm{depth}_R N/IN > 0$. Moreover, if I is $\mathfrak{m}$-primary then M is projective.*

Theorem 6. *Suppose I is weakly $\mathfrak{m}$-full and $\mathrm{depth}_R R/I = 0$. If $\mathrm{Tor}_n^R(M, R/I) = 0$ and $\mathrm{depth}_R(\mathrm{Tor}_{n-1}^R(M, R/I)) > 0$ then $\mathrm{proj.dim}_R M < n - 1$ for all positive integer n.*

These theorems induce the following corollaries.

Corollary 7. [3] *Let R be a local ring and let I be an $\mathfrak{m}$-primary ideal of R. If I is weakly $\mathfrak{m}$-full then R/I is a 1-Tor-test module for projectivity.*

(Proof) The result follows from Theorem 5. □

Corollary 8. [5] *Let R be a local ring and let I be an $\mathfrak{m}$-primary ideal of R. If I is weakly $\mathfrak{m}$-full then R/I and I are strong test modules for projectivity.*

(Proof) By using Corollary 7 and Theorem 4. □

Corollary 9. [4] *Suppose I is weakly $\mathfrak{m}$-full and $\mathrm{depth}_R R/I = 0$, the following statements hold.*

(1) *R/I is a 2-Tor-test module for projectivity.*
(2) *R/I, I and $\Omega_R I$ are 2-test modules for projectivity.*

(Proof) These results follow from Theorem 6 and Theorem 4. □
Finally let me give bad examples.

Example 10. Let k be an algebraically closed field with characteristic zero.

(1) Let R be an artinian complete intersection local ring $k[[x, y]]/(x^2, y^2)$ and put $I = (y)R$. Then, I is $\mathfrak{m}$-primary but not weakly $\mathfrak{m}$-full, and R/I is neither n-Tor-test module for projectivity nor n-test module for projectivity for all positive integer n.

(2) Let R be a one-dimensional complete local hypersurface ring $k[[x,y]]/(xy)$ and put $I=(y^2)R$. Then, I is not $\mathfrak{m}$-primary but weakly $\mathfrak{m}$-full and $\operatorname{depth}_R R/I = 0$. Therefore R/I is both 2-Tor-test module for projectivity and 2-test module for projectivity, and I and $\Omega_R I$ are 2-test modules for projectivity. Nevertheless, $\operatorname{Tor}_1(R/(x), R/I) = \operatorname{Ext}^2(R/(x), R/I) = \operatorname{Ext}^1(R/(x), I) = \operatorname{Ext}^2(R/(x), \Omega_R I) = 0$. This means that R/I is neither 1-Tor-tests module for projectivity nor 1-test module for projectivity, and I and $\Omega_R I$ are not 1-test modules for projectivity.

(3) Let R be a one-dimensional complete intersection local ring $k[[x,y,z]]/(xy,z^2)$ and put $I=(y^2)R, J=(z)R$. Then, I is not weakly $\mathfrak{m}$-full but $\operatorname{depth}_R R/I = 0$ and J is weakly $\mathfrak{m}$-full but $\operatorname{depth}_R(R/J) \neq 0$. For any positive integer i, $\operatorname{Tor}_i(R/I, R/J), \operatorname{Ext}^i(R/J, R/I)$ and $\operatorname{Ext}^i(R/(x), R/J)$ equal to zero. This means that R/I and R/J are neither 2-Tor-tests module for projectivity nor 2-test modules for projectivity. Similarly, I and $\Omega_R I$ are not 2-test modules for projectivity.

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