HOCHSCHILD COHOMOLOGY OF BEILINSON ALGEBRAS OF GRADED DOWN-UP ALGEBRAS

AYAKO ITABA AND KENTA UEYAMA

Abstract. Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $(\deg x, \deg y) = (1, n)$ and $\beta \neq 0$, and let ∇A be the Beilinson algebra of A. If $n = 1$, then a description of the Hochschild cohomology group of ∇A was given by Belmans. In this report, we calculate the Hochschild cohomology group of ∇A for the case $n \geq 2$. Moreover, we apply our results to study the bounded derived category of the noncommutative projective scheme of A.

Keywords: Hochschild cohomology, down-up algebra, Beilinson algebra, derived equivalence.

2010 *Mathematics Subject Classification*: 16E40, 16S38, 16E05, 18E30.

1. Beilinson algebras of graded down-up algebras

In this section, we give a brief overview of the Beilinson algebras of graded down-up algebras. Throughout, let k be an algebraically closed field of $\operatorname{char} k = 0$.

Definition 1 ([1])**.** A connected graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is called a d-dimensional *AS-regular algebra* of Gorenstein parameter l if it satisfies the following conditions:

(i) $\operatorname{gldim} A = d < \infty$,
(ii) $\operatorname{GKdim} A := \inf\{\alpha \in \mathbb{R} \mid \dim_k(\sum_{i=0}^{n} A_i) \leq n^\alpha \text{ for all } n \gg 0\} < \infty$, where $\operatorname{GKdim} A$ is called the *Gelfand-Kirillov dimension* of A, and
(iii) (*Gorenstein condition*) $\operatorname{Ext}^i_A(k, A) \cong \begin{cases} k(l) & (i = d), \\ 0 & (i \neq d). \end{cases}$

For example, if a graded algebra A is commutative, then A is an n-dimensional AS-regular algebra if and only if $A \cong k[x_1, \dots, x_n]$. Also, a graded algebra

$$A = k\langle x, y\rangle/(x^2y + yx^2, xy^2 + y^2x)$$

is a 3-dimensional AS-regular algebra.

Definition 2 ([6])**.** A graded algebra

$$A(\alpha, \beta) := k\langle x, y\rangle/(x^2y - \beta yx^2 - \alpha xyx,\ xy^2 - \beta y^2x - \alpha yxy)$$
$$\deg x = m, \deg y = n \in \mathbb{N}^+$$

with parameters $\alpha, \beta \in k$ is called a *graded down-up algebra*.

The detailed version of this paper has been submitted for publication elsewhere.

The first author was supported by JSPS Grant-in-Aid for Early-Career Scientists 18K13397. The second author was supported by JSPS Grant-in-Aid for Early-Career Scientists 18K13381.

Down-up algebras were originally introduced by Benkart and Roby [6] in the study of the down and up operators on partially ordered sets. Since then, various aspects of these algebras have been investigated. In particular, from the viewpoint of noncommutative projective geometry, the following property is of importance.

Theorem 3 ([11])**.** *Let $A = A(\alpha, \beta)$ be a graded down-up algebra. Then A is a noetherian 3-dimensional AS-regular algebra if and only if $\beta \neq 0$.*

Note that a graded down-up algebra has played a key role as a test case for more complicated situations in noncommutative projective geometry.

Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $\beta \neq 0$, so that A is 3-dimensional AS-regular. Then the Gorenstein parameter ℓ of A is equal to $2(\deg x + \deg y) = 2(m+n)$. The *Beilinson algebra* of A is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}$$

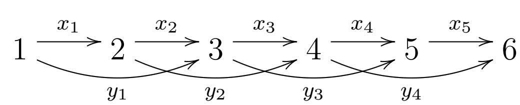
with the multiplication $(a_{ij})(b_{ij}) = \left(\sum_{k=0}^{\ell-1} a_{kj} b_{ik}\right)$. We remark that the Beilinson algebra ∇A of A is a finite-dimensional k-algebra, and it can be given by a quiver with relations. For example, if $\deg x = 1, \deg y = 1$, then ∇A is given by the quiver

$$1 \underset{y_1}{\overset{x_1}{\rightrightarrows}} 2 \underset{y_2}{\overset{x_2}{\rightrightarrows}} 3 \underset{y_3}{\overset{x_3}{\rightrightarrows}} 4$$

(where the Gorenstein parameter of A is $\ell = 2(1+1) = 4$) with relations

$$x_1x_2y_3 - \beta y_1x_2x_3 - \alpha x_1y_2x_3 = 0, \; x_1y_2y_3 - \beta y_1y_2x_3 - \alpha y_1x_2y_3 = 0.$$

Also, if $\deg x = 1, \deg y = 2$, then ∇A is given by the quiver



(where the Gorenstein parameter of A is $\ell = 2(1+2) = 6$) with relations

$$x_1x_2y_3 - \beta y_1x_2x_3 - \alpha x_1y_2x_3 = 0, \; x_2x_3y_4 - \beta y_2x_4x_5 - \alpha x_2y_3x_5 = 0,$$
$$x_1y_2y_4 - \beta y_1y_3x_5 - \alpha y_1x_3y_4 = 0.$$

Let $\mathsf{tails}\, A$ be the quotient category of finitely generated graded right A-modules by the Serre subcategory of finite-dimensional modules, and $\mathsf{mod}\, \nabla A$ the category of finitely generated right ∇A-modules. We remark that $\mathsf{tails}\, A$ is considered as the category of coherent sheaves on the noncommutative projective scheme associated to A in the sense of Artin-Zhang [2]. We write $\mathsf{D^b}(\mathsf{tails}\, A)$ and $\mathsf{D^b}(\mathsf{mod}\, \nabla A)$ for the bounded derived categories of $\mathsf{tails}\, A$ and $\mathsf{mod}\, \nabla A$, respectively.

The following result is obtained as a special case of [13, Theorem 4.14].

Theorem 4. *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $\beta \neq 0$. Then the following statements hold.*

(1) *The Beilinson algebra ∇A of A is an extremely Fano algebra of* $\operatorname{gldim} \nabla A = 2$.

(2) *There exists an equivalence of triangulate categories* $\mathsf{D}^{\mathsf{b}}(\mathsf{tails}\, A) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{mod}\, \nabla A)$.

We note that a Fano algebra was renamed as *an n-representation infinite algebra* in Herschend-Iyama-Oppermann [9] from the viewpoint of higher-dimensional Auslander-Reiten theory. By Theorem 4, the Beilinson algebras of down-up algebras are important not only in noncommutative projective geometry but also in representation theory of finite-dimensional algebras.

2. Hochschild cohomology groups of Beilinson algebras of graded down-up algebras

The aim of this report is to investigate the Hochschild cohomology groups $\mathrm{HH}^i(\nabla A)$ of ∇A of a graded down-up algebra $A = A(\alpha, \beta)$ with $\beta \neq 0$. The i-th *Hochschild cohomology group* $\mathrm{HH}^i(\nabla A)$ of ∇A is defined by

$$\mathrm{HH}^i(\nabla A) := \mathrm{Ext}^i_{(\nabla A)^{\mathrm{e}}}(\nabla A, \nabla A) \ (i \geq 0),$$

where $(\nabla A)^{\mathrm{e}} := (\nabla A)^{\mathrm{op}} \otimes \nabla A$ is the enveloping algebra of ∇A. The family of right $(\nabla A)^{\mathrm{e}}$-modules is one-to-one corresponding to the family of ∇A-bimodules. The low-dimensional Hochschild cohomology groups are described as follows:

- $\mathrm{HH}^0(\nabla A)$ is the center $Z(\nabla A)$ of ∇A.
- $\mathrm{HH}^1(\nabla A)$ is the space of derivations modulo the inner derivation. A derivations is a k-linear map $f : \nabla A \to \nabla A$ such that $f(ab) = af(b) + f(a)b$ for all $a, b \in \nabla A$. A derivation $f : \nabla A \to \nabla A$ is an inner derivation if there is some $x \in \nabla A$ such that $f(a) = ax - xa$ for all $a \in \nabla A$.
- $\mathrm{HH}^2(\nabla A)$ measures the infinitesimal deformations of the algebra ∇A.

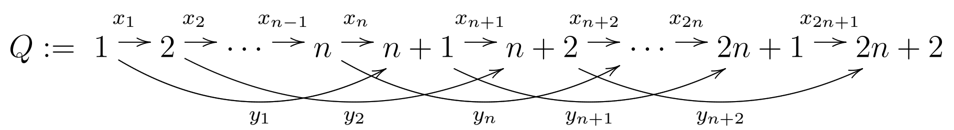
It is known that the Hochschild cohomology of the Beilinson algebra of an AS-regular algebra A is closely related to the Hochschild cohomology of $\mathsf{tails}\, A$ and the infinitesimal deformation theory of $\mathsf{tails}\, A$ (see [12]).

If $\deg x = \deg y = 1$, then a description of $\mathrm{HH}^i(\nabla A)$ has been obtained by Belmans, using a geometric technique.

Theorem 5 ([3, Table 2])**.** *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $\deg x = \deg y = 1$ and $\beta \neq 0$, and ∇A the Beilinson algebra of A. Then the dimension formula of $\mathrm{HH}^i(\nabla A)$ is as follows*:

- $\dim_k \mathrm{HH}^0(\nabla A) = 1$;
- $\dim_k \mathrm{HH}^1(\nabla A) = \begin{cases} 6 & \text{if } \alpha = 0, \\ 3 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta = 0, \\ 1 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$
- $\dim_k \mathrm{HH}^2(\nabla A) = \begin{cases} 9 & \text{if } \alpha = 0, \\ 6 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta = 0, \\ 4 & \text{if } \alpha \neq 0 \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$
- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ *for* $i \geq 3$.

In this report, for $\deg x = 1, \deg y = n \geq 2$, we give the dimension formula of $\mathrm{HH}^i(\nabla A)$. In this case, the Beilinson algebra ∇A is given by the following quiver with relations:



$$f_i := x_i x_{i+1} y_{i+2} - \beta y_i x_{i+n} x_{i+n+1} - \alpha x_i y_{i+1} x_{i+n+1} = 0 \quad (1 \leq i \leq n),$$
$$g := x_1 y_2 y_{n+2} - \beta y_1 y_{n+1} x_{2n+1} - \alpha y_1 x_{n+1} y_{n+2} = 0.$$

The main result of this report is the following theorem.

Theorem 6 ([10, Theorem 1.4]). *Let $A = A(\alpha, \beta)$ be a graded down-up algebra with $\deg x = 1, \deg y = n \geq 2$, and $\beta \neq 0$. We define*

$$\delta_n := \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in k$$

(e.g. $\delta_2 = \alpha^2 + \beta, \delta_3 = \alpha^3 + 2\alpha\beta, \delta_4 = \alpha^4 + 3\alpha^2\beta + \beta^2, \delta_5 = \alpha^5 + 4\alpha^3\beta + 3\alpha\beta^2$). Then the dimension formula of $\mathrm{HH}^i(\nabla A)$ is as follows:

- $\dim_k \mathrm{HH}^0(\nabla A) = 1$;
- $\dim_k \mathrm{HH}^1(\nabla A) = \begin{cases} 4 & \text{if } n \text{ is odd and } \alpha = 0 \text{ (in this case } \delta_n = 0), \\ 3 & \text{if } n \text{ is odd}, \alpha \neq 0, \text{ and } \delta_n = 0, \text{ or if } n \text{ is even and } \delta_n = 0, \\ 2 & \text{if } \alpha^2 + 4\beta = 0 \text{ (in this case } \delta_n \neq 0), \\ 1 & \text{if } \delta_n \neq 0 \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$
- $\dim_k \mathrm{HH}^2(\nabla A) = \begin{cases} 8 & \text{if } n = 2 \text{ and } \delta_2 = 0, \\ 7 & \text{if } n = 2 \text{ and } \alpha^2 + 4\beta = 0 \text{ (in this case } \delta_2 \neq 0), \\ 6 & \text{if } n = 2, \delta_2 \neq 0, \text{ and } \alpha^2 + 4\beta \neq 0, \\ n+5 & \text{if } n \text{ is odd and } \alpha = 0 \text{ (in this case } \delta_n = 0), \\ n+4 & \text{if } n \text{ is odd}, \alpha \neq 0, \text{ and } \delta_n = 0, \text{ or if } n \geq 4 \text{ is even and } \delta_n = 0, \\ n+3 & \text{if } n \geq 3 \text{ and } \alpha^2 + 4\beta = 0 \text{ (in this case } \delta_n \neq 0), \\ n+2 & \text{if } n \geq 3, \delta_n \neq 0, \text{ and } \alpha^2 + 4\beta \neq 0; \end{cases}$
- $\dim_k \mathrm{HH}^i(\nabla A) = 0$ *for* $i \geq 3$.

Remark 7. In the setting of Theorem 6, A is not generated in degree 1, so the geometric approach due to Belmans does not work naively. Our proof of Theorem 6 is purely algebraic by using Green-Snashall's method (see [7] for details).

Recall that Hochschild cohomology is invariant under derived equivalence. Using Theorem 4, Theorem 5, and Theorem 6, we have the following consequence.

Corollary 8 ([10, Corollary 1.5]). *Let $A = A(\alpha, \beta)$ and $A' = A(\alpha', \beta')$ be graded down-up algebras with $\deg x = 1, \deg y = n \geq 1$, where $\beta \neq 0, \beta' \neq 0$. If*

$$\delta_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ \beta & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \delta'_n = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha' & 1 \\ \beta' & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0,$$

then $\mathsf{D^b}(\mathsf{tails}\, A) \not\cong \mathsf{D^b}(\mathsf{tails}\, A')$.

3. Application to the study of Grothendieck groups

In this last section, we apply our results to the study of Grothendieck groups. Let T be a triangulated category, $K_0(\mathsf{T})$ the Grothendieck group of T (see [5, Section 3] for details). If T admits a full strong exceptional sequence of length r, then $K_0(\mathsf{T})$ is $\mathbb{Z}^r$, so $\operatorname{rk} K_0(\mathsf{T}) = r$. If T has the Serre functor S in the sense of Bondal-Kapranov [4], then S induces an automorphism $\mathfrak{s}$ of $K_0(\mathsf{T})$.

Theorem 9 ([3],[5])**.** *Let $\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{coh}} X)$ be the bounded derived category of coherent sheaves on a smooth projective variety X.*

(1) ([5, Lemma 3.1]) *The action of $(-1)^{\dim X}\mathfrak{s}$ on $K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{coh}} X))$ is unipotent.*

(2) ([3, Corollary 25]) *If $\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{coh}} X)$ admits a full strong exceptional sequence, then*

$$\chi(\mathrm{HH}^{\bullet}(X)) = (-1)^{\dim X} \operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{coh}} X)).$$

where $\chi(\mathrm{HH}^{\bullet}(X)) := \sum_{i\in\mathbb{Z}}(-1)^i \dim_k \mathrm{HH}^i(X)$.

Let $A = A(\alpha,\beta)$ be a graded down-up algebra with $\deg x = 1, \deg y = n \geq 1$, and $\beta \neq 0$. Then $\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)$ has a full strong exceptional sequence of length $2n+2$ by [13, Propositions 4.3, 4.4], so $\operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)) = 2n+2$. Moreover $\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)$ has the Serre functor by [14, Appendix A]. Note that $\operatorname{gldim}(\operatorname{\mathsf{tails}} A) = \operatorname{gldim} \nabla A = 2$. If $n = 1$, then $\mathfrak{s}$ acts unipotently on $K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A))$ ([3, comments after Remark 26]), and it follows from Theorem 5 that

$$\chi(\mathrm{HH}^{\bullet}(\nabla A)) = 4 = \operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A))$$

where $\chi(\mathrm{HH}^{\bullet}(\nabla A)) := \sum_{i\in\mathbb{Z}}(-1)^i \dim_k \mathrm{HH}^i(\nabla A)$, so an analogue of Theorem 9 holds. Using Theorem 6 and Happel's trace formula [8, Theorem 2.2], we have the following result.

Proposition 10 ([10, Proposition 3.2])**.** *Let $A = A(\alpha,\beta)$ be a graded down-up algebra with $\deg x = 1, \deg y = n$, and $\beta \neq 0$.*

(1) *If $n = 2$, then $\mathfrak{s}$ acts unipotently on $K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A))$ and*

$$\chi(\mathrm{HH}^{\bullet}(\nabla A)) = 6 = \operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)).$$

(2) *If $n \geq 3$, then $\mathfrak{s}$ does not act unipotently on $K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A))$ and*

$$\chi(\mathrm{HH}^{\bullet}(\nabla A)) = n+2 \neq 2n+2 = \operatorname{rk} K_0(\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)).$$

Remark 11. In respect of Proposition 10, when $n = 2$, $\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)$ behaves a bit like a geometric object (a smooth projective surface), but, when $n \geq 3$, $\mathsf{D}^{\mathsf{b}}(\operatorname{\mathsf{tails}} A)$ is not equivalent to the derived category of any smooth projective surface.

References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension* 3, Adv. Math. **66** (1987), 171–216.

[2] M. Artin and J. J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), no. 2, 228–287.

[3] P. Belmans, *Hochschild cohomology of noncommutative planes and quadrics*, J. Noncommut. Geom. **13** (2019), no. 2, 769–795.

[4] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and reconstructions*, Math. USSR-Izv. **35** (1990), no. 3, 519–541.

[5] A. I. Bondal and A. E. Polishchuk, *Homological properties of associative algebras: the method of helices*, Russian Acad. Sci. Izv. Math. **42** (1994), no. 2, 219–260.
[6] G. Benkart and T. Roby, *Down-up algebras*, J. Algebra **209** (1998), no. 1, 305–344. Addendum: J. Algebra **213** (1999), no. 1, 378.
[7] E. L. Green and N. Snashall, *Projective bimodule resolutions of an algebra and vanishing of the second Hochschild cohomology group*, Forum Math. **16** (2004), 17–36.
[8] D. Happel, *The trace of the Coxeter matrix and Hochschild cohomology*, Linear Algebra Appl. **258** (1997), 169–177.
[9] M. Herschend, O. Iyama, and S. Oppermann, *n-Representation infinite algebras*, Adv. Math. **252** (2014), 292–342.
[10] A. Itaba and K. Ueyama, *Hochschild cohomology related to graded down-up algebras with weights* $(1, n)$, (2019), submitted (arXiv:1904.00677).
[11] E. Kirkman, I. Musson, and D. Passman, *Noetherian down-up algebras*, Proc. Amer. Math. Soc. **127** (1999), no. 11, 3161–3167.
[12] W. Lowen and M. Van den Bergh, Deformation theory of abelian categories, *Trans. Amer. Math. Soc.* **358** (2006), no. 12, 5441–5483.
[13] H. Minamoto and I. Mori, *The structure of AS-Gorenstein algebras*, Adv. Math. **226** (2011), no. 5, 4061–4095.
[14] K. de Naeghel and M. Van den Bergh, *Ideal classes of three-dimensional Sklyanin algebras*, J. Algebra **276** (2004), no. 2, 515–551.

Ayako Itaba
Faculty of Science
Department of Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjyuku-ku, Tokyo, 162-8601, JAPAN
E-mail address: itaba@rs.tus.ac.jp

Kenta Ueyama
Department of Mathematics
Faculty of Education
Hirosaki University
1 Bunkyocho, Hirosaki, Aomori, 036-8560, JAPAN
E-mail address: k-ueyama@hirosaki-u.ac.jp