THE HOCHSCHILD COHOMOLOGY OF A CLASS OF EXCEPTIONAL PERIODIC SELFINJECTIVE ALGEBRAS OF POLYNOMIAL GROWTH

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Abstract. In this paper, we determine the Hochschild cohomology ring of a class of exceptional periodic algebras of polynomial growth.

1. Introduction

This paper is based on joint work with G. Zhou and W. Lyu. It is known that the non-standard periodic representation-infinite selfinjective algebras of polynomial growth are socle deformations of the corresponding periodic standard algebras, and every such an algebra Λ is geometric socle deformation of excactly one representation-infinite standard algebra Λ' of polynomial growth. These algebras Λ and Λ' are called exceptional periodic algebras of polynomial growth in [2]. In [3], their Hochschild cohomology groups $\mathrm{HH}^i(\Lambda)$ and $\mathrm{HH}^i(\Lambda')$ for $i = 0, 1, 2$ are determined, and it is shown that Λ and Λ' are not derived equivalent. However, their Hochschild cohomology groups $\mathrm{HH}^i(\Lambda)$ for $i \geq 3$ and Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ is not computed.

In this paper, we determine the Hochschild cohomology ring of a class of exceptional periodic selfinjective algebras of polynomial growth.

2. Exceptional selfinjective algebras of polynomial growth

In this section, we will explain exceptional selfinjective algebras of polynomial growth in [2]. Let K be an algebraically closed field.

Definition 1. Let A and B be selfinjective K-algebras. Then A and B are *socle equivalent* if $A/\mathrm{soc}\, A$ is isomorphic to $B/\mathrm{soc}\, B$.

Let Λ be a nonstandard representation-infinite selfinjective algebra of polynomial growth over K. Then, there exists a unique standard selfinjective algebra Λ' of tubular type such that

(i) $\dim_K \Lambda = \dim_K \Lambda'$,
(ii) Λ and Λ' are socle equivalent,
(iii) Λ' is degeneration of Λ.

The algebra Λ' is called the *standard form* of Λ. The algebras Λ and Λ' are called the *exceptional selfinjective algebras of polynomial growth*.

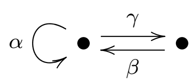
Basic connected selfinjective K-algebras are classified in [1].

The detailed version of this paper will be submitted for publication elsewhere.

Theorem 2 ([1, Theorem 1.1]). *Let Λ be a basic connected selfinjective K-algebra. Then, Λ is socle equivalent to selfinjective algebra of tubular type if and only if exactly one of the following cases holds:*

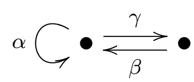
(1) Λ is of tubular type.
(2) $\operatorname{char} K = 3$ and Λ is isomorphic to one of the bound quiver algebras:

Λ_1



$$\alpha^2 = \gamma\beta,\quad \beta\alpha\gamma = \beta\alpha^2\gamma,\quad \alpha^5 = 0$$

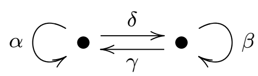
Λ_2



$$\alpha^2\gamma = \beta\alpha^2 = 0,\quad \beta\gamma = \beta\alpha\gamma,\quad \alpha^3 = \gamma\beta$$

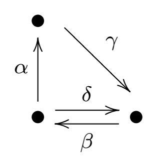
(3) $\operatorname{char} K = 2$ and Λ is isomorphic to one of the bound quiver algebras:

$\Lambda_3(\lambda)$



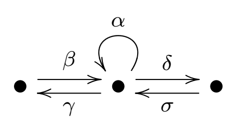
$$\alpha^2 = \sigma\gamma + \alpha^3,\quad \lambda\beta^2 = \gamma\sigma, \gamma\alpha = \beta\gamma,\quad \alpha\sigma = \sigma\beta, \alpha^4 = 0\quad \lambda \in K\backslash\{0,1\}$$

Λ_4



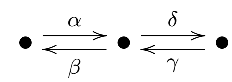
$$\delta\beta\delta = \alpha\gamma,\quad (\beta\delta)^3\beta = 0,\quad \gamma\beta\alpha = \gamma\beta\delta\beta\alpha$$

Λ_5



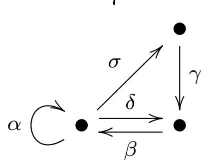
$$\alpha^2 = \gamma\beta, \alpha^3 = \delta\sigma,\quad \beta\delta = 0, \sigma\gamma = 0,\quad \alpha\delta = 0, \sigma\alpha = 0,\quad \beta\gamma = \beta\alpha\gamma$$

Λ_6



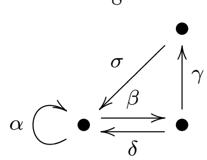
$$\alpha\delta\gamma\delta = 0, \gamma\delta\gamma\beta = 0,\quad \alpha\beta = \alpha\delta\gamma\beta,\quad \beta\alpha = \delta\gamma\delta\gamma$$

Λ_7



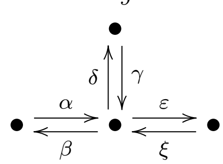
$$\beta\delta = \beta\alpha\delta, \alpha\sigma = 0,\quad \alpha\delta = \sigma\gamma, \gamma\beta\alpha = 0,\quad \alpha^2 = \delta\beta$$

Λ_8



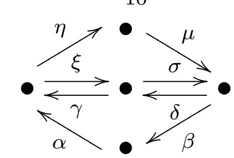
$$\delta\beta = \delta\alpha\beta, \sigma\alpha = 0,\quad \delta\alpha = \gamma\sigma, \alpha\beta\gamma = 0,\quad \alpha^2 = \beta\delta$$

Λ_9



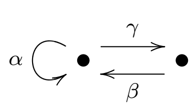
$$\beta\alpha + \varepsilon\xi + \delta\gamma = 0,\quad \alpha\beta = \alpha\delta\gamma\beta,\quad \xi\varepsilon = 0, \gamma\delta = 0$$

Λ_{10}



$$\mu\beta = 0, \alpha\eta = 0,\quad \beta\alpha = \delta\gamma, \eta\mu = \xi\delta,\quad \sigma\delta = \gamma\xi + \sigma\delta\sigma\delta,\quad \delta\sigma\delta\sigma = 0, \xi\gamma\xi\gamma = 0$$

For the algebra Λ_1, the standard form Λ_1' of Λ_1 is given by the following quiver and relations:



$$\alpha^2 = \gamma\beta,\quad \beta\alpha\gamma = 0$$

3. Hochschild cohomology of Λ_1'

In this section, we determine the Hochschild cohomology $\mathrm{HH}^*(\Lambda_1')$ of Λ_1'. In [2], it is shown that Λ_1' is a periodic algbra by giving a periodic projective resolution of Λ_1'. Moreover, by means the projective resolution of Λ_1', the Hochschild cohomology groups $\mathrm{HH}^i(\Lambda_1')$ are determined for $i = 0, 1, 2$ in [3].

By giving the explicit form of the peiodic projective resolution of Λ_1', we determine the the Hochschild cohomology groups of Λ_1'. Moreover, by computing Yoneda product $\mathrm{HH}^i(\Lambda_1') \times \mathrm{HH}^j(\Lambda_1') \to \mathrm{HH}^{i+j}(\Lambda_1')$, we determine the ring structure of Hochschild cohomology $\mathrm{HH}^*(\Lambda_1')$ of Λ_1'.

First, we determine the Hochschild cohomology groups $\mathrm{HH}^i(\Lambda_1')$ of Λ_1'.

Theorem 3. *The Hochschild cohomology groups* $\mathrm{HH}^i(\Lambda_1')$ *of* Λ_1' *are given as follows.*

$$\dim_K \mathrm{HH}^0(\Lambda_1') = 5,$$

$$\dim_K \mathrm{HH}^{6n-5}(\Lambda_1') = \begin{cases} 3 & \text{if } \operatorname{char} K = 2, \\ 4 & \text{if } \operatorname{char} K \neq 2, \end{cases}$$

$$\dim_K \mathrm{HH}^{6n-4}(\Lambda_1') = \dim_K \mathrm{HH}^{6n-3}(\Lambda_1') = \dim_K \mathrm{HH}^{6n-2}(\Lambda_1') = \begin{cases} 4 & \text{if } \operatorname{char} K = 3, \\ 3 & \text{if } \operatorname{char} K \neq 3, \end{cases}$$

$$\dim_K \mathrm{HH}^{6n-1}(\Lambda_1') = \dim_K \mathrm{HH}^{6n}(\Lambda_1') = \begin{cases} 4 & \text{if } \operatorname{char} K = 2, 3, \\ 3 & \text{if } \operatorname{char} K \neq 2, 3, \end{cases}$$

for $n \geq 1$.

Finally, we determine the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda_1')$ of Λ_1' by dividing into the three cases: $\operatorname{char} K = 2$; $\operatorname{char} K = 3$; $\operatorname{char} K \neq 2, 3$.

Theorem 4. *Suppose that* $\operatorname{char} K = 2$. *Then the Hochschild cohomology ring* $\mathrm{HH}^*(\Lambda_1')$ *of* Λ_1' *is given by*

$$\mathrm{HH}^*(\Lambda_1') \cong K[a, b, c, x, p, q, r, y, u, v, z, w]/I,$$

where I *is generated by*

$$\begin{aligned} &a^3, b^2, c^2, ab, ac, bc, cx, a^2x, x^2, ap, bp, cp, aq, bq, cq, ar, br, cr, xq, \\ &ay, by, cy, bv - au, bv - rp, p^2, q^2, pq, qr, r^2, xy, av, cv, a^2v, a^2u, bu, cu, \\ &py, rpx + qy, ry, az, bz, cz - bxv, \\ &y^2, qv, rv - qu, ru, xz, aw - pv, cw - rv, bw - pu, a^2w, \\ &yu, yv - xpu, xpv - qz, pz, rz - yv, qz - axw, bxw - yv, \\ &u^2, yz, v^2 - pw, uv - rw, vz - xpw, uz - yw, z^2 \end{aligned}$$

and the indeterminates of degree are given by

$$\begin{aligned} &|a| = |b| = |c| = 0, |x| = 1, |p| = |q| = |r| = 2, |y| = 3, \\ &|u| = |v| = 4, |z| = 5, |w| = 6. \end{aligned}$$

Theorem 5. *Suppose that* $\operatorname{char} K = 3$. *Then the Hochschild cohomology ring* $\mathrm{HH}^*(\Lambda_1')$ *of* Λ_1' *is given by*

$$\mathrm{HH}^*(\Lambda_1') \cong K[a,b,c,x,y,p,q,z,v,w]/I,$$

where I *is generated by*

$$\begin{aligned}
&a^3, b^2, c^2, ab, ac, bc, cx, a^2x, x^2, bx-ay, cx, by, cy, , y^2, xy, bp, cp, aq, cq, bq-a^2p,\\
&axp-yq, yp+xq, az-xp, bz+xq, a^2z-bz,\\
&ap^2+q^2, xz, yz, av-pq, bv-qp^2, cv, a^2v, qz-xv, apz-yv,\\
&z^2, qv-p^3, pv+aw, bw-qv, cw, zv+xw, yw+p^2z, pqv, qw+v^2
\end{aligned}$$

and the indeterminates of degree are given by

$$\begin{aligned}
&|a|=|b|=|c|=0, |x|=|y|=1, |p|=|q|=2,\\
&|z|=3, |v|=4, |w|=6.
\end{aligned}$$

Theorem 6. *Suppose that* $\operatorname{char} K \neq 2,3$. *Then the Hochschild cohomology ring* $\mathrm{HH}^*(\Lambda_1')$ *of* Λ_1' *is given by*

$$\mathrm{HH}^*(\Lambda_1') \cong K[a,b,c,x,p,q,v,w]/I,$$

where I *is generated by*

$$\begin{aligned}
&a^3, b^2, c^2, ab, ac, bc, cx, a^2x, x^2, bp, cp, aq, bq, cq, a^2p,\\
&ap^2+q^2, pq, av, 2bv+ap^2, cv,\\
&aw+pv, 2bw+qv, cw, a^2w, p^3+qv, qw+v^2
\end{aligned}$$

and the indeterminates of degree are given by

$$|a|=|b|=|c|=0,\ |x|=1,\ |p|=|q|=2,\ |v|=4,\ |w|=6.$$

References

[1] J. Białkowski and A. Skowroński, *Socle deformations of selfinjective algebras of tubular type*, J. Math. Soc. Japan 56, (2004), no. 3, 687–716.

[2] J. Białkowski, K. Erdmann and A. Skowroński, *Periodicity of self-injective algebras of polynomial growth*, J. Algebra 443 (2015), 200–269.

[3] J. Białkowski, K. Erdmann and A. Skowroński, *Hochschild cohomology for periodic algebras of polynomial growth*, J. Pure Appl. Algebra 223 (2017), no. 4, 1548–1589.

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