A BATALIN-VILKOVISKY DIFFERENTIAL ON THE COMPLETE COHOMOLOGY RING OF A FROBENIUS ALGEBRA

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Abstract. We study the existence of a Batalin-Vilkovisky differential on the complete cohomology ring of a Frobenius algebra. We construct a Batalin-Vilkovisky differential on the complete cohomology ring in the case of Frobenius algebras with diagonalizable Nakayama automorphisms.

1. Introduction

Inspired by Buchweitz's result on Tate cohomology of Iwanaga-Gorenstein algebras, Wang has defined *Tate-Hochschild cohomology groups* of an associative algebra A as $\underline{\mathrm{Ext}}^r_{A\otimes_k A^{\mathrm{op}}}(A,A) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A\otimes_k A^{\mathrm{op}})}(A, A[r])$, where $r \in \mathbb{Z}$ and $\mathcal{D}_{\mathrm{sg}}(A\otimes_k A^{\mathrm{op}})$ is the singularity category of $A\otimes_k A^{\mathrm{op}}$. He discovered in [?, ?] that Tate-Hochschild cohomology $\underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A,A) := \bigoplus_{r\in\mathbb{Z}} \underline{\mathrm{Ext}}^r_{A\otimes_k A^{\mathrm{op}}}(A,A)$ has a Gerstenhaber algebra structure. If A is a finite dimensional Frobenius algebra, then the Tate-Hochschild cohomology groups of A are isomorphic to the *complete cohomology groups* $\widehat{\mathrm{HH}}^*(A,A)$ of A, which are the cohomology groups based on a complete resolution of A. Wang also showed that there exists a graded commutative product, called $\star$*-product*, on $\widehat{\mathrm{HH}}^\bullet(A,A)$ such that $\widehat{\mathrm{HH}}^\bullet(A,A)$ is isomorphic to $\underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A,A)$ as graded algebras and that the complete cohomology ring $\widehat{\mathrm{HH}}^\bullet(A,A)$ carries a BV algebra structure in the case that A is a symmetric algebra. In this paper, we generalize Wang's result to the case of finite dimensional Frobenius algebras with diagonalizable Nakayama automorphisms.

Throughout this paper, A denotes a finite dimensional, associative and unital algebra over a field k and let A^{e} be the enveloping algebra $A\otimes_k A^{\mathrm{op}}$ of A. For simplicity, we write $\otimes := \otimes_k, \mathrm{Hom} := \mathrm{Hom}_k$ and $\overline{b}_{1,n} := \overline{b}_1 \otimes \cdots \otimes \overline{b}_n \in \overline{A}^{\otimes n}$ with the quotient vector space $\overline{A} := A/(k\cdot 1_A)$.

2. Preliminaries

Definition 1. A *Gerstenhaber algebra* is a graded k-module $\mathcal{H}^\bullet = \bigoplus_{r\in\mathbb{Z}} \mathcal{H}^r$ equipped with two graded maps, a cup product $\smile\colon \mathcal{H}^m\otimes\mathcal{H}^n \to \mathcal{H}^{m+n}$ of degree 0 and a Lie bracket, called the *Gerstenhaber bracket*, $[\ ,\ ] : \mathcal{H}^m\otimes\mathcal{H}^n\to\mathcal{H}^{m+n-1}$ of degree -1, satisfying

(i) $(\mathcal{H}^\bullet, \smile)$ is a graded commutative algebra with unit $1\in\mathcal{H}^0$.
(ii) $(\mathcal{H}^\bullet[1], [\ ,\ ])$ is a graded Lie algebra with components $(\mathcal{H}^\bullet[1])^r = \mathcal{H}^{r+1}$.
(iii) For homogeneous elements α, β and $\gamma\in\mathcal{H}^\bullet$

$$[\alpha, \beta\smile\gamma] = [\alpha,\beta]\smile\gamma + (-1)^{(|\alpha|-1)|\beta|}\beta\smile[\alpha,\gamma],$$

The detailed version of this paper will be submitted for publication elsewhere.

where $|\alpha|$ denotes the degree of a homogeneous element α in $\mathcal{H}^\bullet$.

Definition 2. A graded commutative algebra $(\mathcal{H}^\bullet = \bigoplus_{r\in\mathbb{Z}} \mathcal{H}^r, \smile)$ with $1 \in \mathcal{H}^0$ is a *Batalin-Vilkovisky algebra* (BV algebra, for short) if there exists a graded k-linear map $\Delta : \mathcal{H}^* \to \mathcal{H}^{*-1}$, called *BV differential*, of degree -1 such that:

(i) $\Delta^2 = 0$ and $\Delta_0(1) = 0$.

(ii) For homogeneous elements α, β and γ in $\mathcal{H}^\bullet$,

$$\begin{aligned}\Delta(\alpha \smile \beta \smile \gamma) = \Delta(\alpha \smile \beta) \smile \gamma + (-1)^{|\alpha|}\alpha \smile \Delta(\beta \smile \gamma)\\ + (-1)^{|\beta|(|\alpha|-1)}\beta \smile \Delta(\alpha \smile \gamma) - \Delta(\alpha) \smile \beta \smile \gamma\\ - (-1)^{|\alpha|}\alpha \smile \Delta(\beta) \smile \gamma - (-1)^{|\alpha|+|\beta|}\alpha \smile \beta \smile \Delta(\gamma),\end{aligned}$$

where $|\alpha|$ denotes the degree of a homogeneous element $\alpha \in \mathcal{H}^\bullet$.

Remark 3. For each BV algebra $(\mathcal{H}^\bullet, \smile, \Delta)$, we can associate a graded Lie bracket $[\ ,\ ]$ of degree -1 as

$$[\alpha, \beta] := (-1)^{|\alpha||\beta|+|\alpha|+|\beta|} \left((-1)^{|\alpha|+1}\Delta(\alpha \smile \beta) + (-1)^{|\alpha|}\Delta(\alpha) \smile \beta + \alpha \smile \Delta(\beta)\right),$$

where α, β are homogeneous elements of $\mathcal{H}^\bullet$. It follows from [**?**, Proposition1.2] that the bracket $[\ ,\ ]$ above makes $(\mathcal{H}^\bullet, \smile, [\ ,\ ])$ into a Gerstenhaber algebra.

Definition 4. Let M be an A-bimodule. We define two complexes $(C^*(A, M), \delta^*)$ and $(C_*(A, M), \partial_*)$ as follows:

$$C^*(A, M) : C^0(A, M) \xrightarrow{\delta^0} C^1(A, M) \to \cdots \to C^n(A, M) \xrightarrow{\delta^n} C^{n+1}(A, M) \to \cdots,$$

where

$$C^0(A, M) := \mathrm{Hom}(k, M) \cong M, \quad C^n(A, M) := \mathrm{Hom}(\overline{A}^{\otimes n}, M),$$

$$\delta^n(f)(\overline{a}_{1,\,n+1}) = a_1 f(\overline{a}_{2,\,n+1}) + \sum_{i=1}^{n}(-1)^i f(\overline{a}_{1,\,i-1} \otimes \overline{a_i a_{i+1}} \otimes \overline{a}_{i+2,\,n}) + (-1)^{n+1} f(\overline{a}_{1,\,n})a_{n+1}.$$

On the other hand,

$$C_*(A, M) : \cdots \to C_n(A, M) \xrightarrow{\partial_n} C_{n-1}(A, M) \to \cdots \to C_1(A, M) \xrightarrow{\partial_1} C_0(A, M),$$

where

$$C_0(A, M) := M, \quad C_n(A, M) := M \otimes \overline{A}^{\otimes n},$$

$$\partial_n(m \otimes \overline{a}_{1,\,n}) = ma_1 \otimes \overline{a}_{2,\,n} + \sum_{i=1}^{n-1}(-1)^i m \otimes \overline{a}_{1,\,i-1} \otimes \overline{a_i a_{i+1}} \otimes \overline{a}_{i+2,\,n} + (-1)^n a_n m \otimes \overline{a}_{1,\,n-1}.$$

The n-th *Hochschild (co)homology group* $\mathrm{H}_*(A, M)$ (resp. $\mathrm{H}^*(A, M)$) of A with coefficients in M is defined by the n-th cohomology group of $C_*(A, M)$ (resp. $C^*(A, M)$).

The *normalized bar resolution* $\mathrm{Bar}(A)$ of A is a projective resolution of A as an A-bimoudle with components $\mathrm{Bar}_n(A) := A \otimes \overline{A}^{\otimes n} \otimes A$ and differentials $d_n : \mathrm{Bar}_n(A) \to$

$\mathrm{Bar}_{n-1}(A)$ given by

$$d_n(a_0 \otimes \overline{a}_{1,n} \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \overline{a}_{1,i-1} \otimes \overline{a_i a_{i+1}} \otimes \overline{a}_{i+2,n} \otimes a_{n+1}.$$

It is easy to check that the two complexes $C^*(A, M)$ and $C_*(A, M)$ are isomorphic to $\mathrm{Hom}_{A^e}(\mathrm{Bar}(A), M)$ and $\mathrm{Bar}(A) \otimes_{A^e} M$. This implies that $\mathrm{H}^*(A, M) \cong \mathrm{Ext}^*_{A^e}(A, M)$ and $\mathrm{H}_*(A, M) \cong \mathrm{Tor}^{A^e}_*(A, M)$.

Definition 5 ([?]). let σ be an automorphism of A. Define a k-linear map

$$B^\sigma_r : C_r(A, A_\sigma) \to C_{r+1}(A, A_\sigma)$$

by

$$B^\sigma_r(a_0 \otimes \overline{a}_{1,r}) = \sum_{i=1}^{r+1} (-1)^{ir} 1 \otimes \overline{a}_i \otimes \cdots \otimes \overline{a}_r \otimes \overline{a}_0 \otimes \overline{\sigma(a_1)} \otimes \cdots \otimes \overline{\sigma(a_{i-1})}.$$

We call B^σ the *Conne operator twisted by* σ and write $B := B^{\mathrm{id}_A}$. Let $T : C_r(A, A_\sigma) \to C_r(A, A_\sigma)$ be the k-linear map defined by

$$T(a_0 \otimes \overline{a}_{1,r}) = \sigma(a_0) \otimes \overline{\sigma(a_1)} \otimes \cdots \otimes \overline{\sigma(a_r)}.$$

A direct calculation shows that $\partial_{r+1} B^\sigma_r - B^\sigma_{r-1}\partial_r = (-1)^{r+1}(\mathrm{id} - T)$ for all $r \geq 0$.

We end this section with recalling Wang's result, which says that Tate-Hochschild cohomology $\underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A, A)$ carries a Gerstenhaber algebra structure.

Theorem 6 ([?, ?]). *There exists a graded map*

$$[\ ,\ ]_{\mathrm{sg}} : \underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A, A) \otimes \underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A, A) \to \underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A, A)$$

of degree -1 *such that the triple* $(\underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A, A), \smile_{\mathrm{sg}}, [\ ,\ ]_{\mathrm{sg}})$ *forms a Gerstenhaber algebra, where* $\smile_{\mathrm{sg}}$ *denotes the Yoneda product.*

3. Complete cohomology of a Frobenius algebra

Recall that a k-algebra A is a *Frobenius algebra* if there exists a non-degenerate bilinear form $\langle -, - \rangle : A \otimes A \to k$ satisfying $\langle ab, c \rangle = \langle a, bc \rangle$ for a, b and $c \in A$. The bilinear form gives rise to a left A-module isomorphism $t : A \to D(A)$ given by $x \mapsto \langle -, x \rangle$, where the left (right) A-module structure of $D(A)$ is given by $(af)(x) := f(xa)$ $((fa)(x) := f(ax))$. For a k-basis $\{u_i\}_i$ of A, we have another k-basis $\{v_i\}_i$ such that $\langle v_i, u_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq r$, where δ_{ij} denotes the Kronecker delta. We call $\{v_i\}_i$ the *dual basis* of $\{u_i\}_i$. It is known that there exists an algebra automorphism ν, the so-called the *Nakayama automorphism*, of A such that $\langle a, b \rangle = \langle b, \nu(a) \rangle$ for $a, b \in A$. Then the Nakayama automorphism ν of A makes the left A-module isomorphism $t : A \to D(A)$ into an A-bimodule isomorphism ${}_1A_\nu \to D(A)$. A Frobenius algebra A is called *symmetric* if the Nakayama automorphism of A is the identity id_A.

Definition 7. Let A be a Frobenius k-algebra.

(1) A *complete resolution* $\mathbf{T}$ of A as an A-bimodule is an exact sequence

$$\mathbf{T}: \cdots \to T_2 \longrightarrow T_1 \xrightarrow{d_1} T_0 \longrightarrow T_{-1} \longrightarrow T_{-2} \to \cdots$$

$$T_0 \searrow A \nearrow T_{-1}$$

where $T_{\geq 0}$ is a projective resolution of A and $T_{<0}$ is a (-1)-shifted injective resolution of A (see [?] for more general cases).

(2) For $r \in \mathbb{Z}$, the r-th *complete cohomology group* of A is defined by the r-th cohomology group of the cochain complex $\operatorname{Hom}_{A^e}(\mathbf{T}, A)$ and denoted by $\widehat{\operatorname{HH}}^r(A, A)$.

Remark that the well-definedness of complete cohomology groups of A follows form [?, Lemma 5.3]. Using the normalized bar resolution $\operatorname{Bar}(A)$, Nakayama [?] constructed the complete bar resolution $\mathbf{X}$ of a Frobenius algebra A defined as follows:

$$\cdots \to X_r \xrightarrow{d_r} X_{r-1} \longrightarrow \cdots \xrightarrow{d_1} X_0 \xrightarrow{d_0} X_{-1} \xrightarrow{d_{-1}} \cdots \longrightarrow X_{-s} \xrightarrow{d_{-s}} X_{-s-1} \to \cdots$$

$$X_0 \xrightarrow{\varepsilon} A \xrightarrow[t]{\sim} {}_1D(A)_{\nu^{-1}} \xrightarrow{D(\varepsilon)} X_{-1}$$

where

$$X_r := \operatorname{Bar}_r(A)\ (r \geq 0), \quad X_{-s} := {}_1D(\operatorname{Bar}_{s-1}(A))_{\nu^{-1}}\ (s \geq 1),$$

$$D(\varepsilon)(f) = f\varepsilon \quad (f \in {}_1D(A)_{\nu^{-1}}), \quad d_0 = D(\varepsilon)t\varepsilon, \qquad d_{-s}(g) = gd_s \quad (g \in X_{-s}).$$

Sanada [?, Lemma1.1] proved that $\operatorname{Hom}_{A^e}(\mathbf{X}, A)$ is isomorphic to the cochain complex

$$\cdots \to C_2(A, {}_1A_{\nu^{-1}}) \xrightarrow{\partial_2} C_1(A, {}_1A_{\nu^{-1}}) \xrightarrow{\partial_1} {}_1A_{\nu^{-1}} \xrightarrow{\mu} A \xrightarrow{\delta^0} C^1(A, A) \xrightarrow{\delta^1} C^2(A, A) \to \cdots,$$

where $\mu : {}_1A_{\nu^{-1}} \to A$ is given by $\mu(x) := \sum_i u_i x v_i$ and A is of degree 0. This complex will be denoted by $(\mathcal{D}^*(A, A), \widehat{d}^*)$. Clearly, we have $\widehat{\operatorname{HH}}^r(A, A) = \operatorname{H}^r(A, A)$ for $r \geq 1$ and $\widehat{\operatorname{HH}}^r(A, A) = \operatorname{H}_{-r-1}(A, {}_1A_{\nu^{-1}})$ for $r \leq -2$.

The following is the product

$$\star : \mathcal{D}^*(A, A) \otimes \mathcal{D}^*(A, A) \to \mathcal{D}^*(A, A)$$

introduced by Wang ([?]): let $f \in C^m(A, A)$, $g \in C^n(A, A)$ and $\alpha = a_0 \otimes \overline{a}_{1,p} \in C_p(A, {}_1A_{\nu^{-1}})$, $\beta = b_0 \otimes \overline{b}_{1,q} \in C_q(A, {}_1A_{\nu^{-1}})$.

(1) $(m, n \geq 0)$ $\star : C^m(A, A) \otimes C^n(A, A) \to C^{m+n}(A, A)$ is given by

$$(f \star g)(\overline{x}_{1,m+n}) := f(\overline{x}_{1,m})g(\overline{x}_{m+1,m+n}) \ ;$$

(2) $(m \geq 0, p \geq 0, p \geq m)$

(a) $\star : C_p(A, {}_1A_{\nu^{-1}}) \otimes C^m(A, A) \to C_{p-m}(A, {}_1A_{\nu^{-1}})$ is given by

$$\alpha \star f := m\alpha(\overline{a}_{1,p}) \otimes \overline{a}_{p+1,r} \ ;$$

(b) $\star : C^m(A, A) \otimes C_p(A, {}_1A_{\nu^{-1}}) \to C_{p-m}(A, {}_1A_{\nu^{-1}})$ is given by

$$f \star \alpha := f(\overline{a}_{p-m+1,p})a_0 \otimes \overline{a}_{1,p-m} \ ;$$

(3) $(m \geq 0, p \geq 0, p < m)$

(a) $\star : C^m(A,A)\otimes C_p(A,{}_1A_{\nu^{-1}})\to C^{m-p-1}(A,A)$ is given by

$$(f\star\alpha)(\overline{x}_{1,\,m-p-1}) := \sum_i f(\overline{x}_{1,\,m-p-1}\otimes\overline{u_i\nu(a_0)}\otimes\overline{a}_{1,\,p})v_i\ ;$$

(b) $\star : C_p(A,{}_1A_{\nu^{-1}})\otimes C^m(A,A)\to C^{m-p-1}(A,A)$ is given by

$$(\alpha\star f)(\overline{x}_{1,\,m-p-1}) := \sum_i u_i\nu(a_0)f(\overline{a}_{1,\,p}\otimes\overline{v}_i\otimes\overline{x}_{1,\,m-p-1})\ ;$$

(4) $(p,q\geq 0)$ $\star : C_p(A,{}_1A_{\nu^{-1}})\otimes C_q(A,{}_1A_{\nu^{-1}})\to C_{p+q+1}(A,{}_1A_{\nu^{-1}})$ is given by

$$\alpha\star\beta := \sum_i v_ib_0\otimes\overline{b}_{1,\,q}\otimes\overline{u_i\nu(a_0)}\otimes\overline{a}_{1,\,p}\ .$$

Proposition 8 ([**?**, Lemma 6.2, Propositions 6.5 and 6.9])**.** *Let A be a Frobenius algebra. Then the product $\star$ is compatible with the differentials $\widehat{d}$ of $\mathcal{D}(A,A)$. Moreover, the induced product on $\widehat{\mathrm{HH}}^\bullet(A,A)$, still denoted by $\star$, is graded commutative and associative. In particular, $(\widehat{\mathrm{HH}}^\bullet(A,A),\star)$ is isomorphic to $(\underline{\mathrm{Ext}}^\bullet_{A^e}(A,A),\smile_{\mathrm{sg}})$ as graded algebras.*

4. Main Result

From now, A denotes a Frobenius k-algebra. Let us recall the result of Wang.

Theorem 9 ([**?**, **?**])**.** *Let A be a symmetric k-algebra. Then the complete cohomology ring $(\widehat{\mathrm{HH}}^\bullet(A,A),\star)$ is a BV algebra together with an operator $\widehat{\Delta}_* : \widehat{\mathrm{HH}}^*(A,A)\to\widehat{\mathrm{HH}}^{*-1}(A,A)$ defined by*

$$\widehat{\Delta}_r = \begin{cases}\Delta_r & \text{if } r\geq 1,\\ 0 & \text{if } r=0,\\ (-1)^r\ B_{-r-1} & \text{if } r\leq -1,\end{cases}$$

*where B_* is the Connes operator, and Δ_* defined in [**?**] is the dual of the Connes operator B_{*-1}. In particular, the induced Gerstenhaber algebra is isomorphic to the one on $\underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A,A)$.*

In the case that A is a symmetric algebra, the Nakayama automorphism ν of A is the identity id_A, and hence $\widehat{\Delta}$ defined on $\mathcal{D}^*(A,A)$ can be always lifted to the homology level. In general, $\widehat{\Delta}$ can not be necessarily defined on the homology level. For this, we use the the subcomplexes of $C^*(A,A)$ and $C_*(A,{}_1A_{\nu^{-1}})$ defined in [**?**]: let Λ be the set of eigenvalues of the Nakayama automorphism ν of A. We assume that $\Lambda\subset k$. Let $\widehat{\Lambda}:=\langle\Lambda\rangle$ be the submonoid of $k^\times$ generated by Λ. For $\lambda\in\Lambda$ with eigenspace A_λ, we write $\overline{A}_\lambda = A_\lambda$ for $\lambda\neq 1$ and $\overline{A}_1 = A_1/(k\cdot 1_A)$ for $\lambda=1$. For any automorphism σ of A and $\mu\in\widehat{\Lambda}$, we define subcomplexes

$$C_r^{(\mu)}(A,{}_1A_\sigma) := \bigoplus_{\mu_i\in\Lambda,\,\prod\mu_i=\mu} A_{\mu_0}\otimes\overline{A}_{\mu_1}\otimes\cdots\otimes\overline{A}_{\mu_r},$$

$$C^r_{(\mu)}(A,A) := \big\{f\in C^r(A,A)\big| f(\overline{A}_{\mu_1}\otimes\cdots\otimes\overline{A}_{\mu_r})\subset A_{\mu\mu_1\cdots\mu_r},\ \text{for any } \mu_i\in\Lambda\big\}.$$

The n-th homology groups of $C_*^{(\mu)}(A, {}_1A_\sigma)$ and $C^*_{(\mu)}(A,A)$ are denoted by $\mathrm{H}_n^{(\mu)}(A, {}_1A_\sigma)$ and $\mathrm{H}^n_{(\mu)}(A,A)$, respectively.

Proposition 10. *For any automorphism σ of A, the restriction of $B^\sigma : C_*(A, {}_1A_\sigma) \to C_{*+1}(A, {}_1A_\sigma)$ to $C_*^{(1)}(A, A_\sigma)$ induces an operator*

$$B^\sigma : \mathrm{H}_*^{(1)}(A, A_\sigma) \to \mathrm{H}_{*+1}^{(1)}(A, A_\sigma),$$

and it satisfies $(B^\sigma)^2 = 0$.

For any $\mu \in \widehat{\Lambda}$, we define a subspace $\mathcal{D}^*_{(\mu)}(A,A)$ of $\mathcal{D}^*(A,A)$ as follows: for any $\mu \in \widehat{\Lambda}$,

$$\mathcal{D}^r_{(\mu)}(A,A) := \begin{cases} C^r_{(\mu)}(A,A) & \text{if } r \geq 0, \\ C^{(\mu)}_{-r-1}(A, {}_1A_\sigma) & \text{if } r \leq -1. \end{cases}$$

One easily check that $\mathcal{D}^*_{(\mu)}(A,A)$ is a subcomplex of $\mathcal{D}^*(A,A)$. We denote $\widehat{\mathrm{HH}}^r_{(\mu)}(A,A) := \mathrm{H}^r(\mathcal{D}^\bullet_{(\mu)}(A,A))$. Using the results of Lambre-Zhou-Zimmermann in [?] yields the following.

Proposition 11. *If the Nakayama automorphism ν of A is diagonalizable, then the following statements hold.*

(1) *For $\mu \neq 1 \in \widehat{\Lambda}$, we get $\widehat{\mathrm{HH}}^*_{(\mu)}(A,A) = 0$.*

(2) *There exists an isomorphism $\widehat{\mathrm{HH}}^*_{(1)}(A,A) \cong \widehat{\mathrm{HH}}^*(A,A)$.*

A direct computation shows that the product $\star$ on $\widehat{\mathrm{HH}}^*(A,A)$ restricts to $\star_{\mu,\mu'}$: $\widehat{\mathrm{HH}}^*_{(\mu)}(A,A) \otimes \widehat{\mathrm{HH}}^*_{(\mu')}(A,A) \to \widehat{\mathrm{HH}}^*_{(\mu\mu')}(A,A)$ for any μ and $\mu' \in \widehat{\Lambda}$. Putting $\star_1 := \star_{1,1}$, we get the following.

Proposition 12. *If the Nakayama automorphism ν of A is diagonalizable, then we have an isomorphism $(\widehat{\mathrm{HH}}^\bullet_{(1)}(A), \star_1) \cong (\widehat{\mathrm{HH}}^\bullet(A), \star)$ of graded algebras.*

We are now ready to prove our main result. Using Lambre-Zhou-Zimmermann's BV differential BV differential Δ^ν on $\mathrm{H}^*_{(1)}(A,A)$ and the twisted Connes operator $B^{\nu^{-1}}$ on $\mathrm{H}_*^{(1)}(A, {}_1A_{\nu^{-1}})$, we have the following.

Theorem 13. *Let A be a Frobenius k-algebra. If the Nakayama automorphism ν is diagonalizable, then the graded commutative ring $(\widehat{\mathrm{HH}}^\bullet_{(1)}(A,A), \star_1)$ is a BV algebra together with an operator $\widehat{\Delta}_* : \widehat{\mathrm{HH}}^*_{(1)}(A,A) \to \widehat{\mathrm{HH}}^{*-1}_{(1)}(A,A)$ defined by*

$$\widehat{\Delta}_r = \begin{cases} \Delta^\nu_r & \text{if } r \geq 1, \\ 0 & \text{if } r = 0, \\ (-1)^i \ B^{\nu^{-1}}_{-r-1} & \text{if } r \leq -1, \end{cases}$$

where $B^{\nu^{-1}}$ is the Connes operator twisted by ν^{-1}, and Δ^ν_* defined in [?] is the dual of the Connes operator B^ν twisted by ν. In particular, the induced Gerstenhaber algebra is isomorphic to the one on $\underline{\mathrm{Ext}}^\bullet_{A\otimes_k A^{\mathrm{op}}}(A,A)$.

Since $\widehat{\mathrm{HH}}^\bullet(A,A) \cong \widehat{\mathrm{HH}}^\bullet_{(1)}(A,A)$ as graded algebras, we have our main result.

Corollary 14. *Let A be a Frobenius k-algebra whose Nakayama automorphism ν is diagonalizable. Then the complete cohomology ring $\widehat{\mathrm{HH}}^{\bullet}(A,A)$ of A is a BV algebra such that the induced Gerstenhaber algebra is isomorphic to the one on $\underline{\mathrm{Ext}}^{\bullet}_{A\otimes_k A^{\mathrm{op}}}(A,A)$.*

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