A CHARACTERIZATION OF LOCAL RINGS OF COUNTABLE REPRESENTATION TYPE

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Abstract. We say that a Cohen–Macaulay local ring has finite CM_+-representation type if there exist only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum. In this article, we consider the converse of an observation of Araya–Iima–Takahashi, which says that local hypersurfaces of countable CM-representation type have finite CM_+-representation type. We give a complete answer in dimension one, and make some observation in higher dimensional cases.

1. Introduction

Cohen–Macaulay representation theory has been studied widely and deeply for more than four decades. Buchweitz, Greuel and Schreyer [2] proved that the local hypersurfaces of finite (resp. countable) CM-representation type, (that is, Cohen–Macaulay local rings possessing finitely/infinitely-but-countably many nonisomorphic indecomposable maximal Cohen–Macaulay modules) are precisely the local hypersurfaces of type (A_n) with $n \geq 1$, (D_n) with $n \geq 4$, and (E_n) with $n = 6, 7, 8$ (resp. (A_∞) and (D_∞)).

In this article, we introduce another representation type, namely, finite CM_+-representation type. We say that a Cohen–Macaulay local ring has finite CM_+-representation type if there exist only finitely many isomorphism classes of indecomposable maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum.

Using the result of Buchweitz, Greuel and Schreyer, Araya, Iima and Takahashi [1] show the following theorem, which provides examples of a Cohen–Macaulay local ring of finite CM_+-representation type.

Theorem 1 (Araya–Iima–Takahashi)**.** *Let R be a complete local hypersurface with uncountable algebraically closed coefficient field of characteristic not two. If R has countable CM-representation type, then R has finite CM_+-representation type.*

Our aim is to consider whether the converse holds or not.

Actually, we see that the converse of the result of Araya–Iima–Takahashi holds in dimension one.

Theorem 2. *Let R be a homomorphic image of a regular local ring. Suppose that R does not have an isolated singularity but is Gorenstein. If $\dim R = 1$, the following are equivalent.*

(1) *The ring R has finite CM_+-representation type.*

This is based on a joint work with J. Lyle and R. Takahashi [5]. The detailed version of this paper will be submitted for publication elsewhere.

(2) *There exist a regular local ring S and a regular system of parameters x, y such that R is isomorphic to $S/(x^2)$ or $S/(x^2y)$.*

When either of these two conditions holds, the ring R has countable CM*-representation type.*

In section 2, we show some properties of rings of finite CM_+-representation type. In section 3, we explain our sketch of the proof of Theorem 2. In section 4, we study rings of finite CM_+-representation type with dimension greater than one.

2. Properties of rings of finite CM_+-representation type

We use the following convention and definitions.

Convention 3. *Throughout this article, unless otherwise specified, we adopt the following convention. Rings are commutative and noetherian, and modules are finitely generated. Subcategories are full and strict (i.e., closed under isomorphism). Subscripts and superscripts are often omitted unless there is a risk of confusion.*

Definition 4. Let R be a ring.

(1) An R-module M is *maximal Cohen–Macaulay* if the inequality $\operatorname{depth} M_\mathfrak{p} \geq \dim R_\mathfrak{p}$ holds for all $\mathfrak{p} \in \operatorname{Spec} R$. Hence, by definition, the zero module is maximal Cohen–Macaulay.

(2) We denote by $\mathsf{mod}\, R$ the category of (finitely generated) R-modules, and by $\mathsf{CM}(R)$ the subcategory of $\mathsf{mod}\, R$ consisting of maximal Cohen–Macaulay R-modules. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$, we denote by $\mathsf{ind}\, \mathcal{X}$ the set of isomorphism classes of indecomposable R-modules in $\mathcal{X}$, and by $\mathsf{add}_R\, \mathcal{X}$ the *additive closure* of $\mathcal{X}$, that is, the subcategory of $\mathsf{mod}\, R$ consisting of direct summands of finite direct sums of objects in $\mathcal{X}$.

(3) A subset S of $\operatorname{Spec} R$ is called *specialization-closed* if $\mathrm{V}(\mathfrak{p}) \subseteq S$ for all $\mathfrak{p} \in S$. This is equivalent to saying that S is a union of closed subsets of $\operatorname{Spec} R$ in the Zariski topology.

(4) Let S be a subset of $\operatorname{Spec} R$. Then it is easy to see that

$$\sup\{\dim R/\mathfrak{p} \mid \mathfrak{p} \in S\} \geq \sup\{n \geq 0 \mid \text{there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } S\},$$

and the equality holds if S is specialization-closed. The *(Krull) dimension* of a specialization-closed subset S of $\operatorname{Spec} R$ is defined as this common number and denoted by $\dim S$.

(5) The *singular locus* of R, denoted by $\operatorname{Sing} R$, is by definition the set of prime ideals $\mathfrak{p}$ of R such that $R_\mathfrak{p}$ is not a regular local ring. It is clear that $\operatorname{Sing} R$ is a specialization-closed subset of $\operatorname{Spec} R$. If R is excellent, then by definition $\operatorname{Sing} R$ is a closed subset of $\operatorname{Spec} R$ in the Zariski topology.

Definition 5. Let R be a Cohen–Macaulay ring. By $\mathsf{CM}_0(R)$ we denote the subcategory of $\mathsf{CM}(R)$ consisting of modules that are locally free on the punctured spectrum of R, and set

$$\mathsf{CM}_+(R) := \mathsf{CM}(R) \setminus \mathsf{CM}_0(R).[1]$$

[1]The index 0 (resp. $+$) in $\mathsf{CM}_0(R)$ (resp. $\mathsf{CM}_+(R)$) means that it consists of modules whose nonfree loci have zero (resp. positive) dimension.

In the rest of this section, we discuss properties of rings of finite CM_+-representation type.

First we consider the Zariski-closedness and dimension of the singular locus in connection with the works of Huneke and Leuschke [3, 4]. They proved in [3] that if R has finite CM-representation type, then it has an isolated singularity, i.e., $\operatorname{Sing} R$ has dimension at most zero. Also, they showed in [4] that if R is complete or has uncountable residue field, and has countable CM-representation type, then $\operatorname{Sing} R$ has dimension at most one. Our result is the following theorem, whose second assertion extends the result of Huneke and Leuschke [4] from countable CM-representation type to countable CM_+-representation type (i.e., having infinitely but countably many nonisomorphic indecomposable maximal Cohen–Macaulay modules that are not locally free on the punctured spectrum).

Theorem 6. *Let $(R, \mathfrak{m}, k)$ be a Cohen–Macaulay local ring.*

1. *Suppose that R has finite CM_+-representation type. Then the singular locus $\operatorname{Sing} R$ is a finite set. Equivalently, it is a closed subset of $\operatorname{Spec} R$ with dimension at most one.*
2. *Suppose that R has countable CM_+-representation type. Then the set $\operatorname{Sing} R$ is at most countable. It has dimension at most one if R is either complete or k is uncountable.*

If R admits a canonical module and has countable CM-representation type, then the localization $R_\mathfrak{p}$ at each prime ideal $\mathfrak{p}$ of R has at most countable CM-representation type as well. This was also shown by Huneke and Leushke [4]. We prove a result on finite CM_+-representation type in the same context.

Theorem 7. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local ring with a canonical module. Suppose that R has finite CM_+-representation type. Then $R_\mathfrak{p}$ has finite CM-representation type for all $\mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}$. In particular, $R_\mathfrak{p}$ has finite CM_+-representation type for all $\mathfrak{p} \in \operatorname{Spec} R$.*

3. Sketch of the proof

In this section, we state a sketch of the proof of Theorem 2.

First we can see that the ring is a hypersurface under the assumption of Theorem 2.

Proposition 8. *Let R be a Gorenstein non-reduced local ring of dimension one. If R has finite CM_+-representation type, then R is a hypersurface.*

By above, we may assume that the considering ring R is hypersurface $S/(f)$, where S is a regular local ring of dimension two. Then we have the following restriction on f.

Proposition 9. *Let $(S, \mathfrak{n})$ be a regular local ring of dimension two. Take an element $0 \neq f \in \mathfrak{n}$ and set $R = S/(f)$. Suppose that R is not an isolated singularity but has finite*

CM_+*-representation type. Then f has one of the following forms:*

$$f = \begin{cases} p^2qr & \text{where } p,q,r \text{ are distinct irreducibles with} \\ & S/(pqr) \text{ having finite } \mathsf{CM}\text{-representation type}, \\ p^2q & \text{where } p \neq q \text{ are irreducibles with} \\ & S/(pq) \text{ having finite } \mathsf{CM}\text{-representation type}, \\ p^2 & \text{where } p \text{ is an irreducible with } S/(p) \text{ having finite } \mathsf{CM}\text{-representation type.} \end{cases}$$

Now we have the following skech of the proof of Theorem 2.

Sketch of the proof of Theorem 2. By the two propositions above, we may assume that R is of the form $S/(f)$ satisfying the conditions in Proposition 9.

Case 1. f is equal to (p^2), where p is an irreducible with $S/(p)$ having finite CM-representation type. In this case, we take any element $t \in \mathfrak{n}$ that is regular on R. We consider the S-algebra $T = S[z]/(tz - p, z^2)$, where z is an indeterminate over S. Then we can see that T is a local complete intersection of dimension 1 and codimension 2 with t being a system of parameters. Moreover, R is naturally embedded in T, and this embedding is a finite birational extension. Here, a ring extension $A \subseteq B$ is called *birational* if $B \subseteq Q(A)$, where $Q(A)$ is the total quotient ring of A.

Now we put the following lemma.

Lemma 10. *Let $A \subseteq B$ be a finite birational extension of 1-dimensional Cohen–Macaulay local rings. Then*
operatorname$\mathsf{ind}\,\mathsf{CM}_+(B)$ *is contained in* $\mathsf{ind}\,\mathsf{CM}_+(A)$.

Applying this lemma to the extension $R \subseteq T$, we have an inclusion $\mathsf{ind}\,\mathsf{CM}_+(T) \subseteq \mathsf{ind}\,\mathsf{CM}_+(R)$. By Proposition 8, $\mathsf{ind}\,\mathsf{CM}_+(T)$ has infinitely many elements. Therefore, R cannot be finite $\mathsf{ind}\,\mathsf{CM}_+$-representation type. This is a contradiction.

Case 2. f is equal to (p^2q), where $p \neq q$ are irreducibles with $S/(pq)$ having finite CM-representation type.

In this case, we use matrices A_i and B_i $(i \geq 1)$, which are constructed as follows. Let x, y be a regular system of parameters of S, namely, $\mathfrak{n} = (x, y)$. Let $h \in \mathfrak{n}^2$ be an irreducible element, and write $h = x^2s + xyt + y^2u$ with $s, t, u \in S$. Let $R = S/(x^2h)$ be a local hypersurface of dimension one. One has $\operatorname{Spec} R = \{\mathfrak{p}, \mathfrak{q}, \mathfrak{m}\}$, where we set $\mathfrak{p} = xR$, $\mathfrak{q} = hR$ and $\mathfrak{m} = \mathfrak{n}R$. For each integer $i \geq 1$ we define matrices

$$A_i = \begin{pmatrix} x & 0 & y^i \\ 0 & xy & x \\ 0 & xh & 0 \end{pmatrix}$$

over S. We put $M_i = \operatorname{Cok}_S A_i$. We can see that M_i are indecomposable object in $\mathsf{CM}_+\,(R)$ and non-isomorphic to each other (we omit the details). This yields that R is of inifinite CM_+ -representation type, a contradiction.

Case 3. f is equal to (p^2qr), where p, q, r are distinct irreducibles with $S/(pqr)$ having finite CM-representation type. In this case, we can use similar argument in Case 2.

□

4. On the higher-dimensional case

In this section, we explore the higher-dimensional case: we consider Cohen–Macaulay local rings R with $\dim R \geq 2$ and having finite CM_+-representation type. We have examples of two-dimensional rings of infinite CM_+-representation type.

Example 11. Let S be a regular local ring with a regular system of parameters x, y, z. Then $R = S/(xyz)$ has infinite CM_+-representation type.

Example 12. Let S be a regular local ring with a regular system of parameters x, y, z. Let

$$f = x^n + x^2ya + y^2b$$

be an irreducible element of S with $n \geq 4$ and $a, b \in S$. Then the hypersurface $R = S/(f)$ has infinite CM_+-representation type.

The following gives an analog of Proposition 8 in 2-dimensional case.

Theorem 13. *Let R be a 2-dimensional henselian Nagata Cohen–Macaulay non-normal local ring. Suppose that R has finite CM_+-representation type. Then the following statements hold.*

1. *There exists a minimal prime $\mathfrak{p}$ of R such that the integral closure $\overline{R/\mathfrak{p}}$ has finite CM-representation type. In particular, if R is a domain, then $\overline{R}$ has finite CM-representation type.*
2. *If R is Gorenstein, then R is a hypersurface.*

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