AN APPLICATION OF A THEOREM OF SHEILA BRENNER FOR HOCHSCHILD EXTENSION ALGEBRAS OF A TRUNCATED QUIVER ALGEBRA

HIDEYUKI KOIE

Abstract. Let A be a truncated quiver algebra over an algebraically closed field such that any oriented cycle in the ordinary quiver of A is zero in A. We give the number of the indecomposable direct summands of the middle term of an almost split sequence for a class of Hochschild extension algebras of A by the standard duality module $D(A)$.

Key Words: Hochschild extension, Hochschild (co)homology, trivial extension, self-injective algebra, almost split sequence, quiver.

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1. Introduction

Let K be an algebraically closed field and $A = K\Delta_A/I$ a bound quiver algebra, where Δ_A is a finite connected quiver and the ideal I is admissible. We denote by $D(A)$ the standard duality module $\mathrm{Hom}_K(A, K)$. By a Hochschild extension over A by $D(A)$, we mean an exact sequence

$$0 \longrightarrow D(A) \xrightarrow{\kappa} T \xrightarrow{\rho} A \longrightarrow 0$$

such that T is a K-algebra, ρ is an algebra epimorphism and κ is a T-bimodule monomorphism. The algebra T is called a Hochschild extension algebra. It is well known that T is isomorphic to $A \oplus D(A)$ with the multiplication

$$(a, f)(b, g) = (ab, ag + fb + \alpha(a, b)),$$

where $\alpha : A \times A \longrightarrow D(A)$ is a 2-cocycle. We denote by $T_\alpha(A)$ the Hochschild extension algebra corresponding to a 2-cocycle α. Then, $T_0(A)$ is just the trivial extension algebra $A \ltimes D(A)$.

In [1], Brenner showed how to determine the number of indecomposable direct summands of the middle term of an almost split sequence starting with a simple module. As a consequence of this result, for a self-injective artin algebra, she obtained the number of indecomposable direct summands of $\mathrm{rad}\, P/\mathrm{soc}\, P$, where P is an indecomposable projective module. These results by Brenner play an important role in the representation theory of algebras. However, in general, it is not easy to compute these numbers for a given algebra. So there is few works to compute these numbers. In [2], Fernández and Platzeck gave a simple interpretation of them in the particular case of the trivial extension $T_0(A)$. This is done by focusing on the number of nonzero cycles in $\Delta_{T_0(A)}$. Fernández and Platzeck proved that the set of nonzero cycles coincides with the set of elementary

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cycles. Using this fact, they gave the numbers considered by Brenner by computing the cardinality of the equivalent classes of the set of nonzero cycles.

In this paper, for a truncated quiver algebra A such that any oriented cycle is zero in A, we give a similar interpretation of the numbers considered by Brenner for a Hochschild extension algebra $T_\alpha(A)$ such that $\Delta_{T_\alpha(A)} = \Delta_{T_0(A)}$ holds. Unfortunately, for a Hochschild extension algebra, the set of nonzero cycles does not coincide with the set of elementary cycles in general. So by defining an *α-revived cycle*, we will prove that a nonzero cycle in $T_\alpha(A)$ is either an elementary cycle or an α-revived cycle. So we enumerate these nonzero cycles and then we can give the numbers considered by Brenner easily.

2. A 2-cocycle induced by a cycle in the ordinary quiver

From now on, let K be an algebraically closed field, Δ a quiver and $A := K\Delta/R_\Delta^n$ $(n \geq 2)$ a truncated quiver algebra such that any oriented cycle in Δ is zero in A. We assume that $\dim A > 1$.

Since A is a truncated quiver algebra, we can take a set $\mathbb{M} := \{p_i \mid i = 1, \dots, t\}$ of paths in Δ such that $\{\overline{p_i} \mid i = 1, \dots, t\}$ is a basis of $\mathrm{soc}_{A^e} A$. Moreover, let $\{\overline{p_1}, \dots, \overline{p_t}, \dots, \overline{p_d}\}$ be a basis of A by taking paths $p_{t+1}, \dots, p_d$ in Δ. We denote by $\{\overline{p_1}^*, \dots, \overline{p_t}^*, \dots, \overline{p_d}^*\}$ the dual basis in $D(A)$. We note that, by [2, Proposition 2.2.], the ordinary quiver $\Delta_{T_0(A)}$ is given by

- $(\Delta_{T_0(A)})_0 = \Delta_0$,
- $(\Delta_{T_0(A)})_1 = \Delta_1 \cup \{y_{p_1}, \dots, y_{p_t}\}$,

where, for each i, y_{p_i} is an arrow from $t(p_i)$ to $s(p_i)$.

Next, under the notation of [3] and [4], we will define a 2-cocycle α. For $n+1 \leq s \leq 2n-2$, let $\gamma = x_1 x_2 \cdots x_s \in \Delta_s^c$ be a cycle. Then it is easy to check that γ is a basic cycle. We regard the subscripts i of x_i modulo s $(1 \leq i \leq s)$. Moreover, $((A \otimes_{A^e} \boldsymbol{P}_*)_s, (\tilde{d}_*)_s)$ is Δ_s^c/C_s-graded and $\{v_i = x_{i+n} \cdots x_{i+s-1} \otimes_{K\Delta_0^e} x_i x_{i+1} \cdots x_{i+n-1} \mid 1 \leq i \leq s\}$ is a basis of $((A \otimes_{K\Delta_0^e} K\Delta_n)_s)_{\overline{\gamma}}$. We denote by $\{v_i^* \mid 1 \leq i \leq s\}$ the dual basis in $D(((A \otimes_{K\Delta_0^e} K\Delta_n)_s)_{\overline{\gamma}})$. Then we have the following complex

$$D(((A \otimes_{K\Delta_0^e} K\Delta_1)_s)_{\overline{\gamma}}) \xrightarrow{0} D(((A \otimes_{K\Delta_0^e} K\Delta_n)_s)_{\overline{\gamma}}) \xrightarrow{D(((\tilde{d}_3)_s)_{\overline{\gamma}})} D(((A \otimes_{K\Delta_0^e} K\Delta_{n+1})_s)_{\overline{\gamma}}),$$

and we have the following isomorphism

$$D(HH_{2,s,\overline{\gamma}}(A)) \cong \mathrm{Ker}\,(D(((\tilde{d}_3)_s)_{\overline{\gamma}})) = \langle v_1^* + \cdots + v_s^* \rangle.$$

We denote the map $\Theta(v_i^*) : A \times A \longrightarrow D(A)$ by α_i for $i = 1, 2, \dots, s$. Then each α_i is the map as follows:

$$\alpha_i(\overline{a}, \overline{b}) = \begin{cases} \overline{x_{i+m} \cdots x_{i+s-1}}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A,\ n \leq m < s \\ & \qquad \text{and } ab = x_i \cdots x_{i+m-1}, \\ \overline{s(x_i)}^* & \text{if } \overline{a}, \overline{b} \neq 0 \text{ in } A \text{ and } ab = x_i \cdots x_{i+s-1}, \\ 0 & \text{otherwise,} \end{cases}$$

where a, b are paths in Δ, m denotes the length of ab. Moreover, $\sum_{i=1}^{s}\alpha_i$ is a 2-cocycle and the cohomology class $[\sum_{i=1}^{s}\alpha_i]$ is a basis of $D(HH_{2,\,s,\,\overline{\gamma}}(A))$. We fix a nonzero element $k(\neq 0)\in K$ and let $\alpha = k\sum_{i=1}^{s}\alpha_i$. Then we have the following proposition.

Proposition 1. *The ordinary quiver of $T_\alpha(A)$ coincides with $\Delta_{T_0(A)}$.*

Proof. We can prove this proposition by a similar way to [3, Theorem 4.3]. □

3. Elementary cycles and α-revived cycles

Let $\alpha = k\sum_{i=1}^{s}\alpha_i$ be the 2-cocycle defined in Section 2. We define an elementary cycle and its weight for $T_\alpha(A)$ based on [2, Definition 3.1]. Let C be an oriented cycle in $\Delta_{T_\alpha(A)}$. We say that C is *elementary* if $C = \delta_2 y_{p_i}\delta_1$ for some paths δ_1 and δ_2 in $K\Delta$ and $p_i\in\mathbb{M}$ such that $\overline{p_i}^*(\overline{\delta_1\delta_2})\neq 0$. Now let $C = a_1\cdots a_j$ be an oriented cycle in $\Delta_{T_\alpha(A)}$ where $a_1,\dots,a_j\in\Delta_1$. We say that C is *α-revived* if there exist a, $b\in\Delta_+$ such that $\overline{a}$, $\overline{b}\neq 0$ in A, $C = a_1\cdots a_j = ab$ and $\alpha(\overline{a},\,\overline{b})\neq 0$. Then, under the notation above, it is easy to see that $j = s$, $C = x_i\cdots x_{i+s-1}$ for some i and $\alpha(\overline{a},\,\overline{b})(1_A) = k$, where k is the fixed element in the above. Moreover, we define a *weight* $w(C)$ of an elementary cycle $C = \delta_2 y_{p_i}\delta_1$ by $\overline{p_i}^*(\overline{\delta_1\delta_2})$, and we also define a *weight* $w(C)$ of an α-revived cycle C by k.

We say that a path q is *contained* in a path q', if $q' = \gamma_1 q\gamma_2$, where γ_1, γ_2 are paths with $t(\gamma_1) = s(q)$ and $s(\gamma_2) = t(q)$.

Remark 2 (cf. [2, Remark 3.3]). If $0\neq\overline{v}\in A$, then there are paths δ_1,δ_2 in $K\Delta$ and $p_j\in\mathbb{M}$ such that $\overline{p_j}^*(\overline{\delta_1 v\delta_2})\neq 0$, and in particular, any nonzero path in A is contained in an elementary cycle.

Remark 3. If $C = a_1\cdots a_m$ with $a_1,\dots,a_m\in(\Delta_{T_\alpha(A)})_1$ is an elementary cycle, then $a_2a_3\cdots a_m a_1$ is also an elementary cycle.

Remark 4. If $C = a_1\cdots a_j$ with $a_1,\dots,a_j\in\Delta_1$ is an α-revived cycle, then $a_2a_3\cdots a_j a_1$ is also an α-revived cycle.

Definition 5 (cf. [2, Definition 3.4])**.** Let q be a path contained in an elementary cycle C of length less than or equal to the length of C. The *supplement* of q in C is defined as follows:

$$\begin{cases}\text{the trivial path } e_{s(q)} & \text{if } s(q)=t(q),\\ \text{the path formed by the remaining arrows of } C & \text{if } s(q)\neq t(q).\end{cases}$$

Theorem 6. *Let C be an oriented cycle in $K\Delta_{T_\alpha(A)}$. Then the following conditions are equivalent:*

(1) *C is an elementary cycle or α-revived cycle.*
(2) *C is nonzero in $T_\alpha(A)$.*

4. An application of a theorem of Brenner

In this section, we give the number of indecomposable direct summands of the middle term of almost split sequence for $T_\alpha(A)$. We define a relation on the set of nonzero oriented cycles with same origin in $\Delta_{T_\alpha(A)}$. We will show that the above number is equal to the cardinality of the equivalence classes.

Definition 7. For each $h \in (\Delta_{T_\alpha(A)})_0$, let us denote by $\mathcal{C}_h$ the set of all oriented cycles C such that $C \neq 0$ in $T_\alpha(A)$ and $s(C) = t(C) = h$. Let C, C' be in $\mathcal{C}_h$. If there exists an arrow a belonging to C and C' with $s(a) = h$ or $t(a) = h$, then we write $C\mathcal{R}C'$.

Definition 8. For each $h \in (\Delta_{T_\alpha(A)})_0$, let $\mathcal{A}_h = \{a \in (\Delta_{T_\alpha(A)})_1 \mid t(a) = h\}$. For a, $a' \in \mathcal{A}_h$, if there exists an arrow $b \in (\Delta_{T_\alpha(A)})_1$ such that $ab \neq 0$ and $a'b \neq 0$ in $T_\alpha(A)$ then we write $a\mathcal{R}'a'$.

We note that, for any path $a \in \mathcal{A}_h$, $a\mathcal{R}'a$ holds.

From now on, we denote by " $\equiv$ " and " $\approx$ " the equivalence relations generated by $\mathcal{R}$ in $\mathcal{C}_h$ and by $\mathcal{R}'$ in $\mathcal{A}_h$, respectively.

Proposition 9. $\mathrm{card}(\mathcal{C}_h/{\equiv}) = \mathrm{card}(\mathcal{A}_h/{\approx})$.

We have the following theorem, which is similar to [2, Proposition 4.9]:

Proposition 10. *Let h be a vertex in $\Delta_{T_\alpha(A)}$, and let e_h be the idempotent element corresponding to h. Then we have $N_{e_h} = n_{e_h} = \mathrm{card}(\mathcal{C}_h/{\equiv})$.*

The following theorems are partial generalizations of [2].

Theorem 11. *Let S_h be the simple $T_\alpha(A)$-module corresponding to the vertex h. Then the number of indecomposable direct summands of the middle term of almost split sequence*

$$0 \longrightarrow S_h \longrightarrow E \longrightarrow \tau^{-1}S_h \longrightarrow 0$$

is equal to the number of equivalence classes in $\mathcal{C}_h$. Furthermore, the number of indecomposable projective summands of E is equal to zero.

Theorem 12. *Let P_h be the indecomposable projective $T_\alpha(A)$-module corresponding to the vertex h. Then the number of indecomposable direct summands of $\mathrm{rad}\, P_h/\mathrm{soc}\, P_h$ is equal to the number of equivalence classes in $\mathcal{C}_h$.*

Corollary 13. *Let $n \geq 3$ and $h \in \Delta_0$ be neither sink nor source in Δ. Then we have $\mathrm{card}(\mathcal{C}_h/{\equiv}) = 1$.*

References

[1] S. Brenner, The almost split sequence starting with a simple module. Arch. Math. **62** (1994) 203–206.

[2] E. Fernández, M. Platzeck, Presentations of trivial extensions of finite dimensional algebras and a theorem of Sheila Brenner, J. Algebra **249** (2002) 326–344.

[3] H. Koie, T. Itagaki, K. Sanada, The ordinary quivers of Hochschild extension algebras for self-injective Nakayama algebras, Communications in Algebra, **46** (2018) No.9, 3950–3964.

[4] H. Koie, T. Itagaki, K. Sanada, *On presentations of Hochschild extension algebras for a class of self-injective Nakayama algebras*, SUT Journal of Mathematics **53** (2017) No.2, 135–148.

Division of General Education
National Institute of Technology (KOSEN) Nagaoka College
888 Nishikatakaimachi Nagaoka Nigata 940-8532 JAPAN

Email address: 1114702@alumni.tus.ac.jp