MUTATIONS FOR STAR-TO-TREE COMPLEXES AND POINTED BRAUER TREES

YUTA KOZAKAI

Abstract. We will give a sequence of irreducible mutations converting Brauer star algebra to any two-restricted star-to-tree complex.

1. Introduction

Throughout this paper, let k be an algebraically closed field, G_0 a Brauer star of type (e, m) and B a Brauer star algebra over k associated to G_0. Moreover modules means finitely generated left module, and the cyclic orderings of Brauer trees are counter clockwise.

Let us begin with the definition of the two-restricted tilting complex for the Brauer star algebra B and the fact on this complex.

Definition 1. [6] Let $\hat{T}$ be a tilting complex over a Brauer star algebra B. We call $\hat{T}$ a two-restricted tilting complex if any indecomposable direct summand of $\hat{T}$ is a shift of the following elementary complex, where the first nonzero term is in degree 0.

- $S_i : 0 \to Q_i \to 0$,
- $T_{jk} : 0 \to Q_j \xrightarrow{h_{jk}} Q_k \to 0$,

where the map h_{jk} has maximal rank among homomorphisms from Q_j to Q_k.

Theorem 2. [6] *There is a one-to-one correspondence between the set of multiplicity-free two-restricted tilting complexes over the Brauer star algebra B and the set of pointed Brauer trees of type (e, m).*

On the other hand, in [2], it is shown that any representation-finite symmetric algebra is tilting-connected. In particular any Brauer tree algebra is a tilting-connected algebra. Hence, for any two-restricted tilting complex $\hat{T}$ over the Brauer star algebra B, there must exist a sequence of irreducible mutation converting B to $\hat{T}$. Regarding this fact, in [7] they give a sequence of irreducible mutation converting B to $\hat{T}$ in the case that $\hat{T}$ corresponds to the pointed Brauer tree with the reverse pointing or the left alternating pointing.

The aim in this paper is, for any two-restricted tilting complex $\hat{T}$, to give an algorithm to find such a sequence of mutations from the pointed Brauer tree to which $\hat{T}$ corresponds.

The detailed version of this paper will be submitted for publication elsewhere.

2. Mutations for Brauer tree algebras

In this section, we recall the tilting mutations and Kauer moves.

Definition-Theorem 3. *([3]) Let Γ be a basic finite dimensional symmetric algebra and T a tilting complex over Γ. For a decomposition $T = M \oplus X$, we take a triangle*

$$X \xrightarrow{f} M' \to Cone(f) \to X[1]$$

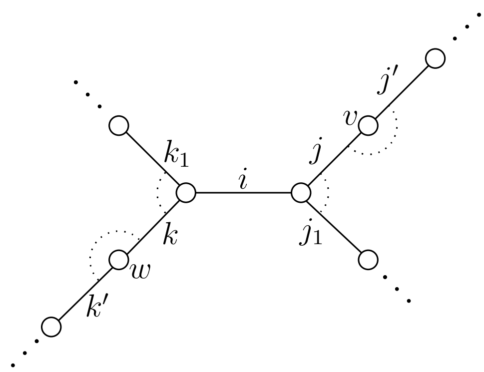
with a minimal left add$M$*-approximation $f : X \to M'$ of X. Then $\mu_X^-(T) := M \oplus Cone(f)$ is a tilting complex again. We call it a left mutation of T with respect to X. Dually we define a right mutation $\mu_X^+(T)$ of T with respect to X. For a left or right mutation $\mu_X^\epsilon(T)$ where $\epsilon \in \{+, -\}$, we call the mutation irreducible if X is an indecomposable complex.*

Remark 4. Without the assumption that Γ is a finite dimensional symmetric algebra, the complex $\mu_X^-(T)$ is not always tilting complex, but is always a silting complex.

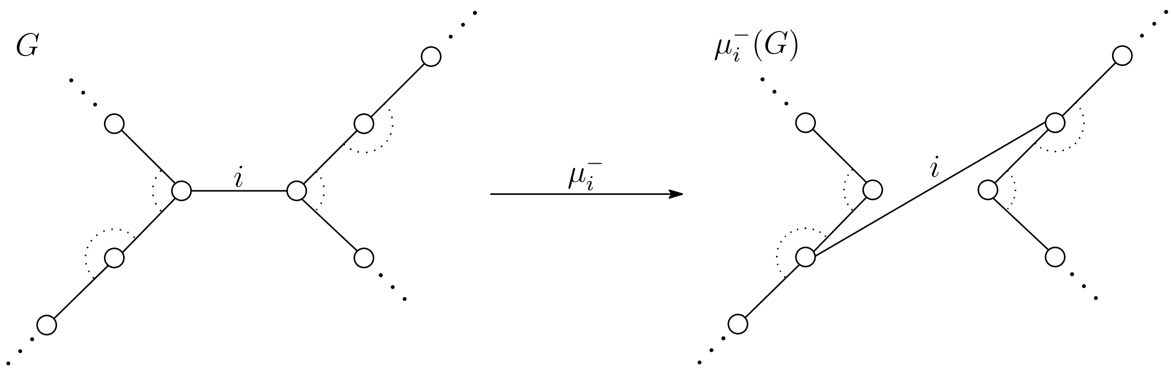
For a tilting complex T over a Brauer tree algebra A, Kauer moves help us to decide the structure of endomorphism algebra of $\mu_X^-(T)$.

Definition 5. (see [5, 1]) Let G be a Brauer tree. For G and an edge i of G, we call a local move as in (i) or (ii) below a Kauer move at i:

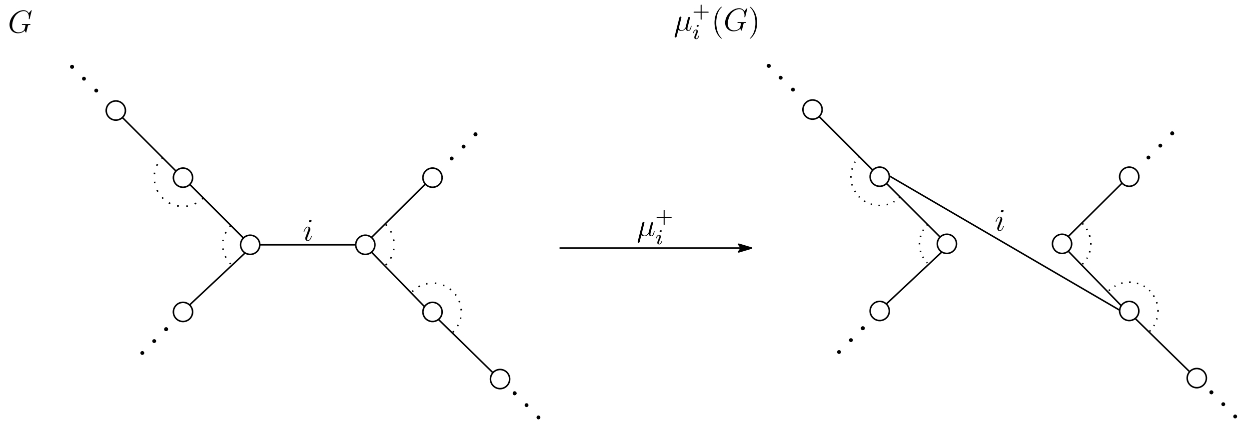
(i) For an edge i of G, let $(j_1, \cdots, j_n = j, i, j_1)$ and $(k_1, \cdots, k_m = k, i, k_1)$ be cyclic orderings of the two vertices adjacent to the edge i (possibly the edge i is an external edge, that is $k_1 = \cdots = k = i$). Let v, w be the vertices of the edges j, k, respectively, which are not adjacent to the edge i. Let j', k' be the next edges before j, k in the cyclic orderings at v, w, respectively.



We define $\mu_i^-(G)$ as follows. Detach i from the two vertices adjacent to the edge i, and attach the edge to v and w so that the cyclic orderings at v and w are $(i, j, \cdots, j', i)$ and $(i, k, \cdots, k', i)$ respectively.



(ii) Dually, we define $\mu_i^+(G)$.



Next result, following from [5, 1], tells us the structure of the opposite algebra of the endomorphism algebra of $\mu_i^-(A)$.

Proposition 6. *([5, 1]) Let Γ be an finite dimensional algebra, and let A_G be a Brauer tree algebra associated to a Brauer tree G. For any i and $\epsilon \in \{+, -\}$, we have an isomorphism* $\mathrm{End}_{D^b(A_G)}(\mu_i^\epsilon(A_G)) \cong A^{op}_{\mu_i^\epsilon(G)}$.

Next, we consider the tilting connectedness for Brauer tree algebras.

Definition 7. Let Γ be a finite dimensional symmetric algebra. Let T_1 and T_2 be basic tilting complexes in $K^b(\Gamma\text{-proj})$. We say that T_1 and T_2 are connected if T_1 can be obtained from T_2 by iterated irreducible mutations. Also $K^b(\Gamma\text{-proj})$ is called tilting-connected if all basic tilting complexes in $K^b(\Gamma\text{-proj})$ are connected to each other.

Theorem 8. *([2]) Let Γ be a finite dimensional symmetric algebra of finite-representation type. Then $K^b(\Gamma\text{-proj})$ is tilting-connected.*

In particular, since Brauer tree algebras are symmetric algebras of finite-representation type, the homotopy category $K^b(A\text{-proj})$ of a Brauer tree algebra A is tilting connected. Hence, for tilting complex T over the Brauer tree algebra A, there is a sequence of mutations

3. The construction of two-restricted star-to-tree complexes from pointed Brauer trees

In this section, we introduce the definition of the pointed Brauer trees, and the construction of two-restricted star-to-tree complexes.

In [6], it was shown that there is a one-to-one correspondence between the set of multiplicity-free two-restricted tilting complexes for the Brauer star algebra of type (e, m) and the set of pointed Brauer trees of type (e, m).

First we give the definition of the pointings and the pointed Brauer trees.

Definition 9. ([6]) A pointing of a Brauer tree consists of the choice of one sector at each exceptional vertex. Then we give a point in that sector for indication. We call the resulting tree with this additional structure a pointed Brauer tree.

Remark 10. For a pointed Brauer tree $G(p)$, we give the one-to-one correspondence among the set of the points, the set of vertices, and the set of the edges as follows. For a point of $G(p)$, let the corresponding vertex be the vertex which the point is on. For an edge of $G(p)$, let the corresponding vertex the farther vertex on the both ends of the edge from the exceptional vertex. We easily see that these correspondences give one-to-one correspondence among the three sets.

In [6], it was shown that there is a one-to-one correspondence between the set of multiplicity-free two-restricted tilting complexes for the Brauer star algebra of type (e, m) and the set of pointed Brauer trees of type (e, m). We give the construction of the two-restricted tilting complexes for the Brauer star algebra based on [6]. To give the construction, we give the definition of the vertex numbering.

Definition 11. ([6]) Let $G(p)$ be a pointed Brauer tree. Then we number each edge in the following way. We call the resulting numbering for the all edges the vertex numbering.

(1) Pick an arbitrary branch at the exceptional vertex as a starting point, and we number the exceptional vertex 0.
(2) Taking Green's walk defined in [4] around the tree in the cyclic ordering, and we assign a number to each vertex whenever the corresponding point is reached.
(3) We number each edge the same number as the corresponding vertex (see Remark 10).

In [6], they introduced an algorithm constructing two-restricted tilting complexes over Brauer star algebras from pointed Brauer trees by using vertex numberings. We explain the algorithm based on [6]. The following algorithm give us a two-restricted star-to-tree complexes from pointed Brauer trees, and this is a one-to-one correspondence between the set of multiplicity-free two-restricted tilting complexes over the Brauer star algebra B and the set of pointed Brauer trees of type (e, m).

Algorithm 12. ([6]) Let $G(p)$ be a pointed Brauer tree of type (e, m). We define a complex $\hat{T}_i$ inductively on the distance from the exceptional vertex as follows, and put $\hat{T} = \bigoplus_{i=1}^{e} \hat{T}_i$. Then $\hat{T}$ is a two-restricted tilting complex over a Brauer star algebra B of type (e, m) with endomorphism algebra the Brauer tree algebra associated to the Brauer tree G.

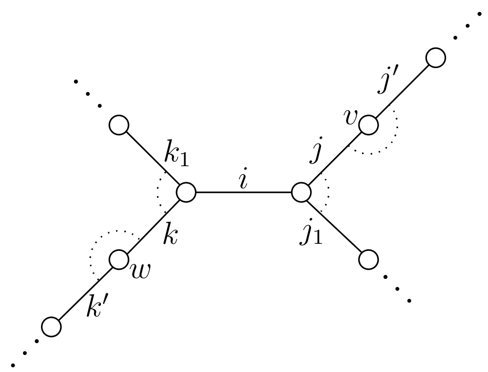
(1) For an edge i adjacent to the exceptional vertex, let $\hat{T}_i$ be the stalk complex $0 \to Q_i \to 0$ where Q_i is in degree 0 and where $B = \bigoplus_{i=1}^{e} Q_i$.

(2) For an edge i not adjacent to the exceptional vertex, let $i_1, i_2, \cdots, i_{n-1}, i_n = i$ be the minimal path from the exceptional vertex to the edge i, and assume that we get $\hat{T}_{i_{n-1}}$. Let $f(i_j)$ be the vertex numbering of i_j for each j. Then we distinguish two cases.
   (2.a) If $f(i_{n-1}) > f(i)$, we set $\hat{T}_i = (0 \to Q_{i_{n-1}} \to Q_i \to 0)[l_n]$, where $[l_n]$ is the shift required to ensure that $Q_{i_{n-1}}$ is in the same degree in $\hat{T}_{i_{n-1}}$ and $\hat{T}_i$.
   (2.b) If $f(i_{n-1}) < f(i)$, we set $\hat{T}_i = (0 \to Q_i \to Q_{i_{n-1}} \to 0)[l_n]$, where again $[l_n]$ is the shift required to ensure that $Q_{i_{n-1}}$ is in the same degree in $\hat{T}_{i_{n-1}}$ and $\hat{T}_i$.

4. Main results

In this section, for a star-to-tree complex $\hat{T}$ corresponding to a pointed Brauer tree $G(p)$, we give an algorithm which give a sequence of irreducible mutations converting B to $\hat{T}$. First we introduce Kauer moves for pointed Brauer trees.

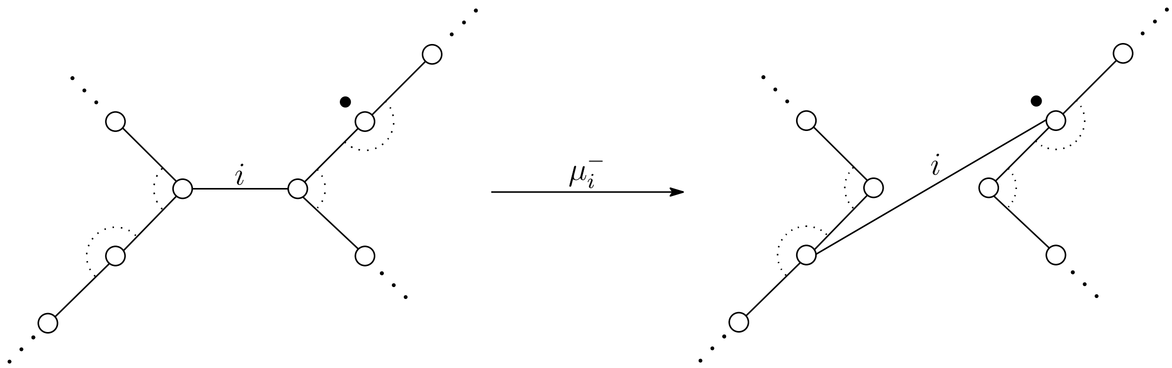
Definition 13. We consider the following situation. Let $G(p)$ be a pointed Brauer tree of a Brauer tree G. For an edge i of $G(p)$, let $(j_1, \cdots, j_n = j, i, j_1)$ and $(k_1, \cdots, k_m = k, i, k_1)$ be cyclic orderings of the two vertices adjacent to the edge i. Let v, w be the vertices of the edges j, k, respectively, which are not adjacent to the edge i. Let j', k' be the next edge before j, k in the cyclic orderings at v, w, respectively.



Then we define a new pointed Brauer tree $\mu_i^-(G(p))$ with the following properties.

(1) As a Brauer tree without the pointing, $\mu_i^-(G(p)) = \mu_i^-(G)$.
(2) (a) When we ignore the edge i, the points at both ends of the edge i in $G(p)$ are in the same sectors as the points in $\mu_i^-(G(p))$.
   (b) Let $r(v)$ be a point on the vertex v.
      (i) If $r(v)$ is in the sector (j, j') in $G(p)$, then $r(v)$ is in the same sector in $\mu_i^-(G(p))$.

(ii) If $r(v)$ is in the sector between (j', j) in $G(p)$, then the point $r(v)$ in $\mu_i^-(G(p))$ is in the sector (j', i) as $G(p)$.



(c) Let $r(w)$ be point on the vertex w. We put the point $r(w)$ in $\mu_i^-(G(p))$ in the same way as we put $r(v)$ in $\mu_i^-(G(p))$.
(d) Any other point in $\mu_i^-(G(p))$ is in the same sector as the point in $G(p)$.

We call this local move Kauer move for the pointed Brauer tree at i.

We will give the algorithm which gives a sequence of irreducible mutations converting B to $\hat{T}$. Let $G(p)$ be a pointed Brauer tree, A_G a Brauer tree algebra associated to G, and T a tilting complex over A_G inducing an inverse derived equivalence to the one induced by $\hat{T}$. To find the required sequence, we enough to give a sequence of irreducible mutations A_G to T. Hence to find such a sequence $(\mu_{i_n}^{\epsilon_n}, \cdots, \mu_{i_2}^{\epsilon_2}, \mu_{i_1}^{\epsilon_1})$ of irreducible mutations, we prepare the following algorithm.

Algorithm 14. Let G be a Brauer tree, and $G(p)$ a pointed Brauer tree of Brauer tree G.

(1) Take an edge corresponding to the first vertex that one would meet on a Green's walk around $G(p)$. If the edge is not adjacent to the exceptional vertex, we take the edge as i_1 and let ϵ_1 be $-$. If the edge is adjacent to the exceptional vertex, then we retake an edge corresponding to the first vertex that one would meet on a *reverse* Green's walk around $G(p)$, and we take the edge as i_1 and let ϵ_1 be $+$.
(2) We take the same process as 1 for the pointed Brauer tree $\mu_{i_1}^{\epsilon_1}(G(p))$ which is defined in Definition 13, and we have the edge i_2 and the sign ϵ_2.
(3) Assume we have a sequence of mutations $(\mu_{i_{l-1}}^{\epsilon_{l-1}}, \cdots, \mu_{i_2}^{\epsilon_2}, \mu_{i_1}^{\epsilon_1})$. Then we take the same process as 1 for the pointed Brauer tree $(\mu_{i_{l-1}}^{\epsilon_{l-1}} \cdots \mu_{i_2}^{\epsilon_2} \mu_{i_1}^{\epsilon_1})(G(p))$, and we get the edge i_l and the sign ϵ_l.
(4) We repeat the process 3 until $(\mu_{i_n}^{\epsilon_n} \cdots \mu_{i_2}^{\epsilon_2} \mu_{i_1}^{\epsilon_1})(G(p))$ gets the Brauer star, and we have a sequence $(\mu_{i_n}^{\epsilon_n}, \cdots, \mu_{i_2}^{\epsilon_2}, \mu_{i_1}^{\epsilon_1})$.

By using a sequence obtained from this algorithm, we can get a sequence of irreducible mutations converting A_G to T corresponding to $G(p)$.

Theorem 15. *Let G be a Brauer tree, $G(p)$ a pointed Brauer tree of the Brauer tree G, and A_G a Brauer tree algebra associated to G. Moreover let $\hat{T}$ be a two-restricted star-to-tree complex corresponding to the pointed Brauer tree $G(p)$, T a tilting complex over A_G*

inducing an inverse derived equivalence to the one induced by $\hat{T}$. Then the sequence of irreducible mutations obtained from Algorithm 14 converts A_G to T.

Moreover, on the Kauer moves for pointed Brauer trees, we get the following theorem which will be helpful to decide the structures of endomorphism algebras of two-restricted star-to-tree complexes.

Theorem 16. *Let G be a Brauer tree, $G(p)$ a pointed Brauer tree of G, $\mu_i^\epsilon(G(p))$ a pointed Brauer tree obtained by applying the Kauer move for pointed Brauer trees where $\epsilon \in \{+, -\}$, and $\hat{T}(G(p))$ a two-restricted star-to-tree complex corresponding to $G(p)$. Assume that the sum of all distance of the edges of $\mu_i^\epsilon(G(p))$ from the exceptional vertex is strictly smaller than that of $G(p)$. Then the star-to-tree complex obtained by applying the mutation μ_i^ϵ to $\hat{T}(G(p))$ is isomorphic to the star-to-tree complex corresponding to $\mu_i^\epsilon(G(p))$. That is, we get the following isomorphism:*

$$\mu_i^\epsilon(\hat{T}(G(p))) \cong \hat{T}(\mu_i^\epsilon(G(p))).$$

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Faculty of Architecture
Shibaura Institute of Technology (part-time lecturer)
3-7-5, Toyosu, Koto-ku, Tokyo, 135-8548 JAPAN
Email address: `1115702@alumni.tus.ac.jp`