NONCOMMUTATIVE MATRIX FACTORIZATIONS AND NONCOMMUTATIVE KNÖRRER'S PERIODICITY

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Abstract. In commutative ring theory, Knörrer's periodicity theorem is a powerful tool to study Cohen-Macaulay representation theory over hypersurfaces, and matrix factorizations are essential ingredients to prove the theorem. In order to study noncommutative hypersurfaces, which are major objects of study in noncommutative algebraic geometry, we introduce a notion of noncommutative matrix factorization and show noncommutative graded versions of Eisenbud's matrix factorization theorem and Knörrer's periodicity theorem. Furthermore, we give four graphical methods to compute the stable category of graded maximal Cohen-Macaulay modules over a skew quadric hypersurface.

1. Introduction

This article is based on our works [4] and [5].

Let $S = k[[x_1, \dots, x_n]]$ be the formal power series ring in n variables over an algebraically closed field k of characteristic not equal to 2, and let $f \in (x_1, \dots, x_n)^2 \subset S$ be a nonzero element. A matrix factorization of f is a pair (Φ, Ψ) of $r \times r$ square matrices whose entries are elements in S such that $\Phi\Psi = \Psi\Phi = fE_r$. In [2, Section 6], Eisenbud showed the factor category $\underline{\mathrm{MF}}_S(f) := \mathrm{MF}_S(f)/\operatorname{add}\{(1, f), (f, 1)\}$ of the category $\mathrm{MF}_S(f)$ of matrix factorizations of f is equivalent to the stable category $\underline{\mathrm{CM}}(S/(f))$ of maximal Cohen-Macaulay $S/(f)$-modules. By this equivalence, we can apply the theory of (reduced) matrix factorizations to the representation theory of Cohen-Macaulay modules (with no free summand) over hypersurfaces. In [3, Theorem 3.1], Knörrer proved the following famous theorem, which is now called Knörrer's periodicity theorem.

Theorem 1 ([3]). *Let $S = k[[x_1, \dots, x_n]]$ and $0 \neq f \in (x_1, \dots, x_n)^2$. Then*

$$\underline{\mathrm{CM}}(S/(f)) \cong \underline{\mathrm{MF}}_S(f) \cong \underline{\mathrm{MF}}_{S[[u,v]]}(f + u^2 + v^2) \cong \underline{\mathrm{CM}}(S[[u, v]]/(f + u^2 + v^2)).$$

In commutative ring theory, Knörrer's periodicity theorem is a powerful tool to study Cohen-Macaulay representation theory. In this article, we discuss what happens if we replace S in the above theorem by an AS-regular algebra, which is a noncommutative graded analogue of a regular local ring in noncommutative algebraic geometry.

2. Preliminaries

2.1. Noncommutative Hypersurfaces.

Throughout this paper, we fix an algebraically closed field k of characteristic not equal to 2.

The detailed version of this paper will be submitted for publication elsewhere.

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The following classes of algebras are main objects of study in noncommutative algebraic geometry.

Definition 2. A noetherian connected graded algebra $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ is called an *AS-regular algebra* (resp. *AS-Gorenstein algebra*) of dimension n if

(1) $\operatorname{gldim} S = n < \infty$ (resp. $\operatorname{injdim}_S S = \operatorname{injdim}_{S^o} S = n < \infty$), and

(2) $\operatorname{Ext}^i_S(k, S) \cong \operatorname{Ext}^i_{S^o}(k, S) \cong \begin{cases} 0 & \text{if } i \neq n \\ k & \text{if } i = n \end{cases}$

where S^o is the the opposite ring of S.

A *quantum polynomial algebra* of dimension n is a noetherian AS-regular algebra A of dimension n with $H_S(t) := \sum_{i\in\mathbb{Z}}(\dim_k S_i)t^i = (1-t)^{-n}$.

Definition 3. Let S be a ring.

(1) $f \in S$ is called regular if, for every $a \in S$, $af = 0$ or $fa = 0$ implies $a = 0$.
(2) $f \in S$ is called normal if $Sf = fS$.

Note that S is a domain if and only if every non-zero element is regular. Moreover a central element is normal, so if S is commutative, then every element is normal.

If S is a quantum polynomial algebra of dimension n and $f \in S$ is a homogeneous regular normal element of positive degree d, then $A = S/(f)$ is a noetherian AS-Gorenstein algebra of dimension $n-1$, and A is regarded as (a homogeneous coordinate ring of) a noncommutative hypersurface of degree d.

2.2. **Totally Reflexive Modules.** Let $\mathcal{C}$ be an additive category and $\mathcal{S}$ a set of objects of $\mathcal{C}$ closed under direct sums. Then the *factor category* $\mathcal{C}/\mathcal{S}$ has the same objects as $\mathcal{C}$ and the morphism space is given by $\operatorname{Hom}_{\mathcal{C}/\mathcal{S}}(M, N) = \operatorname{Hom}_{\mathcal{C}}(M, N)/\mathcal{S}(M, N)$ where $\mathcal{S}(M, N)$ consists of all morphisms from M to N that factor through objects in $\mathcal{S}$. Note that $\mathcal{C}/\mathcal{S}$ is also an additive category.

Definition 4. Let A be a (graded) ring. A (graded) right A-module M is called *totally reflexive* if

(1) $\operatorname{Ext}^i_A(M, A) = 0$ for all $i \geq 1$,
(2) $\operatorname{Ext}^i_{A^o}(\operatorname{Hom}_A(M, A), A) = 0$ for all $i \geq 1$, and
(3) the natural biduality map $M \to \operatorname{Hom}_{A^o}(\operatorname{Hom}_A(M, A), A)$ is an isomorphism.

The category consisting of finitely generated totally reflexive modules is denoted by $\operatorname{TR}(A)$. (The category consisting of finitely generated graded totally reflexive modules is denoted by $\operatorname{TR}^{\mathbb{Z}}(A)$.)

Let $\mathcal{P}$ be a set of finitely generated (graded) free right A-modules. Then the stable category of $\operatorname{TR}(A)$ is defined by $\underline{\operatorname{TR}}(A) := \operatorname{TR}(A)/\mathcal{P}$. (The stable category of $\operatorname{TR}^{\mathbb{Z}}(A)$ is defined by $\underline{\operatorname{TR}}^{\mathbb{Z}}(A) := \operatorname{TR}^{\mathbb{Z}}(A)/\mathcal{P}$.)

If A is a noetherian AS-Gorenstein algebra, then

$$\operatorname{CM}^{\mathbb{Z}}(A) := \{\text{finitely generated graded modules } M \text{ such that } \operatorname{Ext}^i_A(M, A) = 0 \text{ for all } i \geq 1\}$$

coincides with $\operatorname{TR}^{\mathbb{Z}}(A)$, and hence $\underline{\operatorname{CM}}^{\mathbb{Z}}(A) := \operatorname{CM}^{\mathbb{Z}}(A)/\mathcal{P}$ coincides with $\underline{\operatorname{TR}}^{\mathbb{Z}}(A)$.

3. Noncommutative Matrix Factorizations

Definition 5 ([4])**.** Let S be a ring and $f \in S$ an element. A *noncommutative right matrix factorization* of f over S is a sequence of right S-module homomorphisms $\{\phi^i : F^{i+1} \to F^i\}_{i\in\mathbb{Z}}$ where F^i are free right S-modules of rank r for some $r \in \mathbb{N}$ such that there is a commutative diagram

$$\begin{CD} F^{i+2} @>{\phi^i\phi^{i+1}}>> F^i \\ @V{\cong}VV @VV{\cong}V \\ S^r @>{f\cdot}>> S^r \end{CD}$$

for every $i \in \mathbb{Z}$. A morphism $\mu : \{\phi^i : F^{i+1} \to F^i\}_{i\in\mathbb{Z}} \to \{\psi^i : G^{i+1} \to G^i\}_{i\in\mathbb{Z}}$ of noncommutative right matrix factorizations is a sequence of right S-module homomorphisms $\{\mu^i : F^i \to G^i\}_{i\in\mathbb{Z}}$ such that the diagram

$$\begin{CD} F^{i+1} @>{\phi^i}>> F^i \\ @V{\mu^{i+1}}VV @VV{\mu^i}V \\ G^{i+1} @>{\psi^i}>> G^i \end{CD}$$

commutes for every $i \in \mathbb{Z}$. We denote by $\mathrm{NMF}_S(f)$ the category of noncommutative right matrix factorizations.

Let S be a graded ring and $f \in S_d$ a homogeneous element. A *noncommutative graded right matrix factorization* of f over S is a sequence of graded right S-module homomorphisms $\{\phi^i : F^{i+1} \to F^i\}_{i\in\mathbb{Z}}$ where F^i are graded free right S-modules of rank r for some $r \in \mathbb{N}$ such that there is a commutative diagram

$$\begin{array}{ccc} F^{i+2} & \xrightarrow{\qquad\qquad\phi^i\phi^{i+1}\qquad\qquad} & F^i \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \bigoplus_{s=1}^r S(-m_{i+2,s}) = \bigoplus_{s=1}^r S(-m_{is}-d) & \xrightarrow{f\cdot} & \bigoplus_{s=1}^r S(-m_{is}) \end{array}$$

for every $i \in \mathbb{Z}$. We can similarly define the category of noncommutative graded right matrix factorizations $\mathrm{NMF}_S^{\mathbb{Z}}(f)$.

Remark 6. Let S be a (graded) ring and $f \in S$ a (homogeneous) element.

(1) Let $\{\phi^i : F^{i+1} \to F^i\}_{i\in\mathbb{Z}}$ be a noncommutative right matrix factorization of f over S of rank r. We often identify $F^i = S^r$. In this case, every ϕ^i is the left multiplication of a matrix Φ^i whose entries are elements in S, so that $\Phi^i\Phi^{i+1} = fE_r$ where E_r is the identity matrix of size r.

(2) Let $\{\phi^i : F^{i+1} \to F^i\}_{i\in\mathbb{Z}}$ be a noncommutative graded right matrix factorization of f over S of rank r such that $F^i = \bigoplus_{s=1}^r S(-m_{is})$. In this case, we may write $\phi^i = (\phi^i_{st})$ where $\phi^i_{st} : S(-m_{i+1,t}) \to S(-m_{is})$ is the left multiplication of an element in $S_{m_{i+1,t}-m_{is}}$, so ϕ^i is the left multiplication of a matrix Φ^i whose entries are homogeneous elements in S, so that $\Phi^i\Phi^{i+1} = fE_r$ where E_r is the identity matrix of size r.

Definition 7 ([4])**.** Let S be a ring and $f \in S$. For a free right S-module F, we define $\phi_F, {}_F\phi \in \mathrm{NMF}_S(f)$ by

$$\begin{aligned} \phi_F^{2i} &= \mathrm{id}_F : F \to F, \quad \phi_F^{2i+1} = f\cdot : F \to F, \\ {}_F\phi^{2i} &= f\cdot : F \to F, \quad {}_F\phi^{2i+1} = \mathrm{id}_F : F \to F. \end{aligned}$$

We define $\mathcal{F} := \{\phi_F \mid F \in \operatorname{mod} S \text{ is free}\}$, $\mathcal{G} := \{\phi_F \oplus {}_G\phi \mid F, G \in \operatorname{mod} S \text{ are free}\}$ and $\underline{\mathrm{NMF}}_S(f) := \mathrm{NMF}_S(f)/\mathcal{G}$.

Let S be a graded ring and $f \in S_d$. For a graded free right S-module F, we define $\phi_F, {}_F\phi \in \mathrm{NMF}_S^{\mathbb{Z}}(f)$ by

$$\begin{aligned} \phi_F^{2i} &= \mathrm{id}_F : F(-id) \to F(-id), & \phi_F^{2i+1} &= f\cdot : F(-id-d) \to F(-id), \\ {}_F\phi^{2i} &= f\cdot : F(-id-d) \to F(-id), & {}_F\phi^{2i+1} &= \mathrm{id}_F : F(-id-d) \to F(-id-d). \end{aligned}$$

We define $\mathcal{F} := \{\phi_F \mid F \in \operatorname{grmod} S \text{ is free}\}$, $\mathcal{G} := \{\phi_F \oplus {}_G\phi \mid F, G \in \operatorname{grmod} S \text{ are free}\}$ and $\underline{\mathrm{NMF}}_S^{\mathbb{Z}}(f) := \mathrm{NMF}_S^{\mathbb{Z}}(f)/\mathcal{G}$.

Theorem 8 ([4])**.** *If S is a noetherian ring, $f \in S$ is a regular normal element, and $A = S/(f)$, then there are fully faithful embeddings*

$$\mathrm{NMF}_S(f)/\mathcal{F} \to \mathrm{TR}(A) \quad \text{and} \quad \underline{\mathrm{NMF}}_S(f) \to \underline{\mathrm{TR}}(A)$$

A similar result holds in the graded case.

The following theorem is a noncommutative graded version of Eisenbud's matrix factorization theorem.

Theorem 9 ([1], [4])**.** *Let S be a graded quotient algebra of a noetherian AS-regular algebra and $f \in S_d$ a regular normal element, and $A = S/(f)$. Then*

$$\mathrm{NMF}_S^{\mathbb{Z}}(f)/\mathcal{F} \cong \mathrm{TR}_S^{\mathbb{Z}}(A) \quad \text{and} \quad \underline{\mathrm{NMF}}_S^{\mathbb{Z}}(f) \cong \underline{\mathrm{TR}}_S^{\mathbb{Z}}(A)$$

where $\mathrm{TR}_S^{\mathbb{Z}}(A) := \{M \in \mathrm{TR}^{\mathbb{Z}}(A) \mid \operatorname{projdim}_S M < \infty\}$ and $\underline{\mathrm{TR}}_S^{\mathbb{Z}}(A) := \mathrm{TR}_S^{\mathbb{Z}}(A)/\mathcal{P}$. In particular, if S is a noetherian AS-regular algebra, then

$$\mathrm{NMF}_S^{\mathbb{Z}}(f)/\mathcal{F} \cong \mathrm{CM}^{\mathbb{Z}}(A) \quad \text{and} \quad \underline{\mathrm{NMF}}_S^{\mathbb{Z}}(f) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A).$$

4. Knörrer's Periodicity Theorem

Definition 10. Let S be a (graded) ring and σ a (graded) ring automorphism of S. An *Ore extension* $S[u;\sigma]$ of S by σ is a (graded) ring such that $S[u;\sigma] = S[u]$ as a (graded) free right S-module, and $au = u\sigma(a)$ for $a \in S$.

Theorem 11 ([5])**.** *Let S be a noetherian ring, and $f \in S$ a regular normal element. If σ, τ are ring automorphisms of S such that $\sigma(f) = \tau(f) = f$ and $af = f\sigma(\tau(a)) = f\tau(\sigma(a))$ for every $a \in S$, then there is a fully faithful embedding*

$$\underline{\mathrm{NMF}}_S(f) \to \underline{\mathrm{NMF}}_{S[u;\sigma][v;\tau]}(f + uv).$$

A similar result holds in the graded case.

The first main result of this article is the following theorem, which is a noncommutative graded version of Knörrer's periodicity theorem.

Theorem 12 ([5]). *Let S be a noetherian AS-regular algebra and $f \in S$ a regular normal homogeneous element of even degree. If there exists a graded algebra automorphism σ such that $\sigma(f) = f$ and $af = f\sigma^2(a)$ for every $a \in S$, then*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathrm{NMF}}^{\mathbb{Z}}_S(f) \cong \underline{\mathrm{NMF}}^{\mathbb{Z}}_{S[u;\sigma][v;\sigma]}(f+u^2+v^2) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(S[u;\sigma][v;\sigma]/(f+u^2+v^2)).$$

where $\deg u = \deg v = \frac{1}{2}\deg f$.

Note that the technical assumptions in Theorem 12 are needed to guarantee $f+u^2+v^2 \in S[u;\sigma][v;\sigma]$ is a homogeneous normal element. If $f \in S$ is a regular central homogeneous element of even degree, then we may take $\sigma = \mathrm{id}_S$ to apply Theorem 12.

Theorem 12 is a useful tool to compute $\underline{\mathrm{CM}}^{\mathbb{Z}}(S/(f))$ over a noncommutative hypersurface $S/(f)$ since it reduces the number of variables. If f is a central element of degree 2, then there is another way to reduce the number of variables, which applies only in the noncommutative setting.

Theorem 13. *If S is a quantum polynomial algebra and $f \in S_2$ is a regular central element, then*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S[u;-1][v;-1]/(f+u^2+v^2)) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(S[u;-1]/(f+u^2)) \times \underline{\mathrm{CM}}^{\mathbb{Z}}(S[v;-1]/(f+v^2)).$$

Example 14. If $S = k\langle x,y,z\rangle/(yz+zy, zx+xz, xy+yx) = k[x][y;-1][z;-1]$ and $A = S/(x^2+y^2+z^2)$, then

$$\begin{aligned}\underline{\mathrm{CM}}^{\mathbb{Z}}(A) &\cong \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x][y;-1]/(x^2+y^2)) \times \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x][z;-1]/(x^2+z^2))\\ &\cong \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x,y]/(x^2+y^2)) \times \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x,z]/(x^2+z^2))\\ &\cong \mathcal{D}^b(\mathrm{mod}\, k^2) \times \mathcal{D}^b(\mathrm{mod}\, k^2) \cong \mathcal{D}^b(\mathrm{mod}\, k^4).\end{aligned}$$

5. Knörrer's Periodicity for Skew Quadric Hypersurfaces

It is well-known that A is the homogeneous coordinate ring of a smooth quadric hypersurface in $\mathbb{P}^{n-1}$ if and only if $A \cong k[x_1,\dots,x_n]/(x_1^2+\cdots+x_n^2)$. Applying the graded Knörrer's periodicity theorem (see also Theorem 12), we have

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \begin{cases}\underline{\mathrm{CM}}^{\mathbb{Z}}(k[x_1]/(x_1^2)) \cong \mathcal{D}^b(\mathrm{mod}\, k) & \text{if } n \text{ is odd,}\\ \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x_1,x_2]/(x_1^2+x_2^2)) \cong \mathcal{D}^b(\mathrm{mod}\, k^2) & \text{if } n \text{ is even.}\end{cases}$$

In this section, we study a skew version of this equivalence using graphical methods.

A *graph* G consists of a set $V(G)$ of vertices and a set $E(G)$ of edges between two vertices. In this paper, we assume that every graph has no loop and there is at most one edge between two distinct vertices. An edge between two vertices $v, w \in V(E)$ is written by $(v,w) \in E(G)$.

Notation 15. For a symmetric matrix $\varepsilon := (\varepsilon_{ij}) \in M_n(k)$ such that $\varepsilon_{ii} = 1$ and $\varepsilon_{ij} = \varepsilon_{ji} = \pm 1$, we fix the following notations:

1. the standard graded algebra $S_\varepsilon := k\langle x_1,\dots,x_n\rangle/(x_ix_j - \varepsilon_{ij}x_jx_i)$, called a (± 1)-*skew polynomial algebra* in n variables,
2. the point scheme E_ε of S_ε,
3. the central element $f_\varepsilon := x_1^2+\cdots+x_n^2 \in S_\varepsilon$,
4. $A_\varepsilon := S_\varepsilon/(f_\varepsilon)$, and

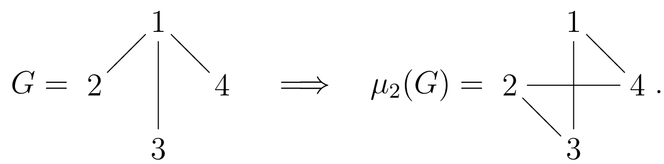
(5) the graph G_ε where $V(G_\varepsilon) = \{1, \ldots, n\}$ and $E(G_\varepsilon) = \{(i,j) \mid \varepsilon_{ij} = \varepsilon_{ji} = 1\}$.

Let G be a graph. A graph G' is a *full subgraph* of G if $V(G') \subset V(G)$ and $E(G') = \{(v,w) \in E(G) \mid v,w \in V(G')\}$. For a subset $I \subset V(G)$, we denote by $G \setminus I$ the full subgraph of G such that $V(G \setminus I) = V(G) \setminus I$. For a full subgraph G' of G, we define the *complement graph* of G' in G by $G \setminus G' := G \setminus V(G')$.

Definition 16 (Mutation [5])**.** Let G be a graph and $v \in V(G)$. The *mutation* $\mu_v(G)$ of G at v is a graph $\mu_v(G)$ where $V(\mu_v(G)) = V(G)$ and

$$E(\mu_v(G)) = \{(v,u) \mid (v,u) \notin E(G), u \neq v\} \cup \{(u,u') \mid (u,u') \in E(G), u,u' \neq v\}.$$

Example 17.



Lemma 18 (Mutation Lemma [5])**.** *If $G_{\varepsilon'} = \mu_v(G_\varepsilon)$ for some $v \in V(G_\varepsilon)$, then $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Definition 19 (Relative Mutation [5])**.** Let $v, w \in V(G)$ be distinct vertices. Then the *relative mutation* $\mu_{v \leftarrow w}(G)$ of G at v with respect to w is a graph $\mu_{v \leftarrow w}(G)$ where $V(\mu_{v \leftarrow w}(G)) = V(G)$ and $E(\mu_{v \leftarrow w}(G))$ is given by the following rules:

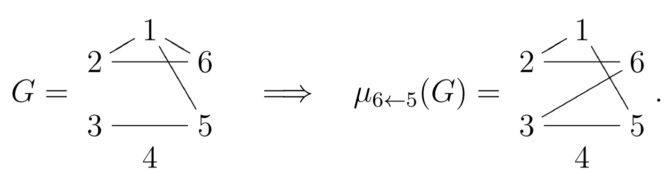
(1) For distinct vertices $u, u' \neq v$, we define that $(u,u') \in E(\mu_{v \leftarrow w}(G)) :\Leftrightarrow (u,u') \in E(G)$.

(2) For a vertex $u \neq v, w$, we define that

$$(v,u) \in E(\mu_{v \leftarrow w}(G)) :\Leftrightarrow \begin{cases} (v,u) \in E(G) \text{ and } (w,u) \notin E(G), \text{or} \\ (v,u) \notin E(G) \text{ and } (w,u) \in E(G). \end{cases}$$

(3) We define that $(v,w) \in E(\mu_{v \leftarrow w}(G)) :\Leftrightarrow (v,w) \in E(G)$.

Example 20.



Lemma 21 (Relative Mutation Lemma [5])**.** *Suppose that $u \in V(G_\varepsilon)$ is an isolated vertex. If $G_{\varepsilon'} = \mu_{v \leftarrow w}(G_\varepsilon)$ for some distinct $v, w \in V(G_\varepsilon)$ not equal to u, then $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Definition 22. An *isolated segment* $[v,w]$ of a graph G consists of distinct vertices $v, w \in V(G)$ with an edge $(v,w) \in E(G)$ between them such that neither v nor w are connected by an edge to any other vertex.

Lemma 23 (Knörrer's Reduction [5])**.** *Suppose that $[v,w]$ is an isolated segment in G_ε. If $G_{\varepsilon'} = G_\varepsilon \setminus \{v,w\}$, then $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Note that Knörrer's reduction is a consequence of Theorem 12.

Lemma 24 (Two Points Reduction [5])**.** *Suppose that $v, w \in V(G_\varepsilon)$ are two distinct isolated vertices. If $G_{\varepsilon'} = G_\varepsilon \setminus \{v\}$, then $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A_{\varepsilon'})^{\times 2}$.*

Note that two points reduction is a consequence of Theorem 13.

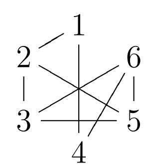
The second main result of this article is the following theorem, which shows that the four graphical operations are very powerful in computing $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)$.

Theorem 25 ([5])**.** *By using mutation, relative mutation, Knörrer reduction, and two points reduction, we can completely compute $\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon)$ up to $n = 6$.*

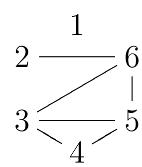
Example 26. Let $S_\varepsilon = k\langle x_1, \dots, x_6\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)$ where

$$\varepsilon_{12} = \varepsilon_{14} = \varepsilon_{23} = \varepsilon_{25} = \varepsilon_{35} = \varepsilon_{36} = \varepsilon_{46} = \varepsilon_{56} = +1,$$
$$\varepsilon_{13} = \varepsilon_{15} = \varepsilon_{16} = \varepsilon_{24} = \varepsilon_{26} = \varepsilon_{34} = \varepsilon_{45} = -1.$$

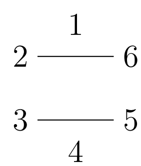
Let $A_\varepsilon = S_\varepsilon/(f_\varepsilon)$ where $f_\varepsilon = x_1^2 + \cdots + x_6^2 \in S_\varepsilon$. Then

$G_\varepsilon =$  .

One can check that

$G' := \mu_4\mu_2(G_\varepsilon) =$ 

and

$\mu_{4\leftarrow 5}\mu_{4\leftarrow 3}\mu_{3\leftarrow 2}\mu_{5\leftarrow 2}(G') =$  .

Hence, by using Knörrer's reduction and two points reduction, we have

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x]/(x^2))^{\times 2} \cong \mathcal{D}^b(\operatorname{mod} k)^{\times 2} \cong \mathcal{D}^b(\operatorname{mod} k^2).$$

By using Theorem 25, we can obtain the following result.

Theorem 27 ([5])**.** *Let ℓ be the number of irreducible components of E_ε that are isomorphic to $\mathbb{P}^1$. Assume that $n \le 6$.*

(1) *If n is odd, then $\ell \le 10$ and*

$$\ell = 0 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathcal{D}^b(\operatorname{mod} k),$$
$$0 < \ell \le 3 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathcal{D}^b(\operatorname{mod} k^4),$$
$$3 < \ell \le 10 \iff \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathcal{D}^b(\operatorname{mod} k^{16}).$$

(2) *If n is even, then $\ell \leq 15$ and*

$$0 \leq \ell \leq 1 \Longleftrightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathcal{D}^b(\operatorname{mod} k^2),$$
$$1 < \ell \leq 6 \Longleftrightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathcal{D}^b(\operatorname{mod} k^8),$$
$$6 < \ell \leq 15 \Longleftrightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathcal{D}^b(\operatorname{mod} k^{32}).$$

In particular, [6, Conjecture 1.3] *holds true for $n \leq 6$.*

Remark 28. It is known that [6, Conjecture 1.3] fails in the case $n = 7$ (see [5, Remark 6.21]).

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