PURE DERIVED CATEGORIES AND WEAK BALANCED BIG COHEN–MACAULAY MODULES

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Abstract. We report a new approach to reach the pure derived category of flat modules over a commutative noetherian ring of finite Krull dimension. Using it, we concretely connect two different stable categories over a Gorenstein ring; the first one is the stable category of Gorenstein-projective modules, and the other is the stable category of Gorenstein-flat cotorsion modules. Although they are triangulated equivalent, we report that the latter has some advantage in terms of pure-injectivity. This advantage along with the notion of weak balanced big Cohen–Macaulay modules naturally leads us to an infinite version of Cohen-Macaulay representation theory.

1. Introduction

A specialist of model theory of modules, Gena Puninski [18] proposed an interesting study of Cohen–Macaulay representations via pure-injectivity. His idea was based on importance of infinitely generated pure-injective modules over artinian rings. As shown by Tachikawa [21, Corollary 9.5] and Auslander [2, Theorem A], a (possibly non-commutative) artinian ring $A$ is of finite representation type if and only if any indecomposable pure-injective module is finitely generated. (See also Auslander [1, Corollary 4.8], Ringel and Tachikawa [19, Corollary 4.4] and Prest [17, §5.3.4].) This fact implies that if $A$ is not of finite representation type, then there exists an infinitely generated pure-injective $A$-module which is indecomposable. Some of such modules have a role to control behavior of finitely generated modules, see Crawley-Boevey [5]. See also Benson and Krause [3] for importance of pure-injective modules in their context.

The next computation is given by Puninski [18].

Example 1. Let $k$ be an algebraic closed field with char $k 
eq 2$, and set $R = k[[x, y]]/(x^2)$. Indecomposable infinitely generated pure-injective $R$-modules $M$ with Hom$_R(k, M) = 0$ are just $R_{(x)}, xR_{(x)}$ and $\bar{R}$ up to isomorphism, where $\bar{R}$ is the integral closure of $R$ in the total quotient ring $R_{(x)}$.

Let $(R, \mathfrak{m}, k)$ be a CM (Cohen–Macaulay) local ring. Puninski meant by a CM $R$-module an $R$-module $M$ such that Ext$^i_R(k, M) = 0$ for $i < \dim R$. To avoid confusion, let us express such modules as “CM” modules. The above example computes all indecomposable infinitely generated “CM” modules over $k[[x, y]]/(x^2)$ such that they are pure-injective.

Unlike artinian rings, $R$ having positive Krull dimension easily admits (trivial) indecomposable pure-injective modules which are infinitely generated. The localization of $R$ at any minimal prime ideal is such an $R$-module, and it is always a “CM” $R$-module.

The detailed version of this paper will be submitted for publication elsewhere.
Moreover, if \( \dim R > 1 \), then any indecomposable injective module corresponding to a high-one prime ideal becomes a “CM” \( R \)-module, see [18, Remark 10.1]. Therefore we cannot simply extend the result of Tachikawa and Auslander. Actually, Puninski [18, Question 10.2] left the next question:

Let \( R \) be the formal power series ring in two variables over an algebraic closed field, and let \( \mathfrak{m} \) be its maximal ideal. Is any indecomposable pure-injective “CM” \( R \)-module \( M \) with \( M/\mathfrak{m}M \neq 0 \) finitely generated?

Although the last condition can avoid a lot of infinitely generated modules, this question might be still unreasonable. One reason is that the vanishing condition of \( \text{Ext}^i_R(k,M) \) for \( i < \dim R \) need not provide the smallest closed subset of the Ziegler spectrum of \( R \) containing all indecomposable finitely generated maximal CM modules up to isomorphism. A better class of modules is formed by weak balanced big CM modules in the sense of Holm [9]. His results [9, Theorem B and Proposition 2.4] can imply that the class of these modules corresponds to the smallest closed subset. Moreover, Puninski’s “CM” modules agree with Holm’s weak balanced big CM modules if \( R \) has at most dimension one. Hence we can naturally continue Puninski’s work.

However, the smallest closed subset still contains kind of trivial ones; indecomposable flat cotorsion modules. In fact, the closed subset is occupied by them if and if \( R \) is regular. Then, in terms of singularity theory, it is very natural to remove them. Sections 2 and 3 are devoted to explain that there is a natural stable category of modules to study the rest part of the closed subset.

2. The pure derived category of flat modules

The stable category of maximal CM modules is a fundamental tool in CM representation theory. It is constructed by identifying two maps whose difference factors through some finitely generated projective module. Over a Gorenstein local ring \( R \), this stable category is triangulated equivalent to the homotopy category \( K_{\text{ac}}(\text{proj} \ R) \) of acyclic complexes of finitely generated projective modules. Then, the larger category \( K_{\text{ac}}(\text{Proj} \ R) \) formed by acyclic complexes of arbitrary projective modules and its corresponding stable category consisting of Gorenstein-projective modules look natural places to discuss infinitely generated CM representations. However, focusing on only “modulo projective modules” could lose an important viewpoint of pure-injectivity. This is just because, projective modules need not be pure-injective. In this section, passing through the pure derived category of flat modules, we will arrive at another stable category, which preserves pure-injectivity nicely, and extends the stable category of maximal CM modules, see Section 3.

Let us start with an arbitrary ring \( A \). A complex \( X \) of left \( A \)-modules is said to be pure acyclic if it is acyclic (i.e. exact) and \( M \otimes_A X \) is acyclic for any right \( A \)-modules. The homotopy category \( K(\text{Flat} \ A) \) of complexes of flat left \( A \)-modules has a full subcategory \( K_{\text{pac}}(\text{Flat} \ A) \) consisting of pure acyclic complexes. The pure derived category \( D(\text{Flat} \ A) \) of flat \( A \)-modules is defined as the Verdier quotient category \( K(\text{Flat} \ A)/K_{\text{pac}}(\text{Flat} \ A) \). This category appeared in Neeman’s work [16], see also Murfet and Salarian [12]. Neeman proved that the canonical composition \( K(\text{Proj} \ A) \to K(\text{Flat} \ A) \to D(\text{Flat} \ A) \) is a triangulated equivalence.
A left $A$-module $M$ is said to be \textit{cotorsion} if $\text{Ext}^i_A(F,M) = 0$ for any flat left $A$-module and any positive integer $i$. The category of cotorsion left $A$-modules is denoted by $\text{Cot} A$. Moreover, we set $\text{FlCot} A = \text{Flat} A \cap \text{Cot} A$; its objects are called flat cotorsion $A$-modules. Štovíček’s [20, Corollary 5.8] refining Gillespie’s [7, Corollary 4.10] implies that the canonical composition $K(\text{FlCot} A) \to K(\text{Flat} A) \to D(\text{Flat} A)$ is a triangulated equivalence. Consequently, it holds that

$$K(\text{Proj} A) \cong D(\text{Flat} A) \cong K(\text{FlCot} A).$$

Replacement of a complex $X \in K(\text{Proj} A)$ with $Y \in K(\text{FlCot} A)$ is given by a pure quasi-isomorphism $X \to Y$, that is, a quasi-isomorphism whose mapping cone is pure acyclic. However, we are not able to understand this replacement in detail by the general theory.

A complex $X$ of projective (resp. flat cotorsion) left $A$-modules is said to be \textit{totally acyclic} if it is acyclic and $\text{Hom}_A(X,F)$ is acyclic for any projective (flat cotorsion) left $A$-module $F$. Moreover, a complex $X$ of flat left $A$-modules is said to be $F$-\textit{totally acyclic} if it is acyclic and $E \otimes_A X$ is acyclic for any injective right $A$-module $E$. We denote by $K_{\text{tac}}(\text{Proj} A)$ and $K_{\text{tac}}(\text{FlCot} A)$ the full subcategories of $K(\text{Proj} A)$ and $K(\text{FlCot} A)$ formed by totally acyclic complexes respectively. Moreover, let $D_{\text{Ftac}}(\text{Flat} A)$ be the full subcategory of $D(\text{Flat} A)$ formed by $F$-totally acyclic complexes.

Suppose that $A$ is a right coherent ring and any flat left $A$-module has finite projective dimension. Then any $F$-totally acyclic complex of projective left $A$-modules becomes totally acyclic. Similarly, any $F$-totally acyclic complex of flat cotorsion left $A$-modules becomes totally acyclic. Therefore, restricting the above triangulated equivalences to $F$-totally acyclic complexes, we get

$$K_{\text{tac}}(\text{Proj} A) \cong D_{\text{Ftac}}(\text{Flat} A) \cong K_{\text{tac}}(\text{FlCot} A).$$

A left $A$-module is called \textit{Gorenstein-projective} if it is the kernel of the $0$th differential map of some totally acyclic complex of projective modules. The category of Gorenstein-projective modules is denoted by $\text{GProj} A$. It is well-known that this category has a Frobenius structure naturally, and hence its stable category $\text{GProj} A$ modulo projective modules becomes a triangulated category which is triangulated equivalent to $K_{\text{tac}}(\text{Proj} A)$.

A left $A$-module is called \textit{Gorenstein-flat} if it is the kernel of the $0$th differential map of some $F$-totally acyclic complex of flat modules. The category of Gorenstein-flat modules is denoted by $\text{GFlat} A$. Its stable category modulo flat modules could be too huge, hence we take a smaller category

$$\text{GFlCot} A := \text{GFlat} A \cap \text{Cot} A,$$

whose objects are Gorenstein-flat cotorsion modules. As observed by Gillespie [8] essentially, this category also has a Frobenius structure, and its stable category $\text{GFlCot} A$ modulo flat cotorsion modules is triangulated equivalent to $K_{\text{tac}}(\text{FlCot} A)$. Therefore we have

$$\text{GProj} A \cong K_{\text{tac}}(\text{Proj} A) \cong K_{\text{tac}}(\text{FlCot} A) \cong \text{GFlCot} A.$$
Henceforth, \( R \) denotes a commutative noetherian ring with finite Krull dimension \( d \). We would like to provide another approach to reach the pure derived category of flat modules over \( R \). Set \( U = \text{Spec} \, R \), and \( U_i = \{ p \in U \mid \text{dim} \, R/p = i \} \). Consider the canonical morphism \( \text{id}_{\text{Mod} \, R} \rightarrow \bar{\lambda}^{U_i} = \prod_{p \in U_i} \Lambda^p(R_p \otimes_R -) \), where \( \Lambda^p \) stands for the p-adic completion functor \( \lim_{\leftarrow n \geq 1} (R/p \otimes_R -) \) on the category \( \text{Mod} \, R \) of \( R \)-modules. Using a standard construction of Čech complexes along with the natural transformations \( \text{id}_{\text{Mod} \, R} \rightarrow \bar{\lambda}^{U_i} \), we get a complex of functors:

\[
L^U = \left( \prod_{0 \leq i \leq d} \bar{\lambda}^{U_i} \rightarrow \prod_{0 \leq i < j \leq d} \bar{\lambda}^{U_j} \bar{\lambda}^{U_i} \rightarrow \cdots \rightarrow \bar{\lambda}^{U_d} \ldots \bar{\lambda}^{U_0} \right),
\]

where \( U \) stands for the family \( \{ U_i \}_{0 \leq i \leq d} \). To a complex \( X \) of \( R \)-modules, \( L^U \) naturally assigns a double complex \( L^U X \) with a canonical morphism \( X \rightarrow L^U X \). Taking their total complexes, we get a natural chain map \( \ell^U X : X \rightarrow \text{tot} L^U X \). This construction appeared in the previous work [13] with Yoshino.

For simplicity, let us write \( \lambda^U = \text{tot} L^U \), which is an endofunctor on the category \( C(\text{Mod} \, R) \) of complexes. This can be restricted to an endofunctor on the category \( C(\text{Flat} \, R) \) of complexes of flat modules. More precisely, it factors through the inclusion from the category \( C(\text{FlCot} \, R) \) of complexes of flat cotorsion modules into \( C(\text{Flat} \, R) \):

\[
C(\text{Flat} \, R) \xrightarrow{\lambda^U} C(\text{FlCot} \, R) \xrightarrow{\text{inc}} C(\text{Flat} \, R).
\]

The above sequence naturally induces a sequence of triangulated functors on homotopy categories:

\[
K(\text{Flat} \, R) \xrightarrow{\lambda^U} K(\text{FlCot} \, R) \xrightarrow{\text{inc}} K(\text{Flat} \, R).
\]

Theorem 2. The triangulated functor \( \lambda^U : K(\text{Flat} \, R) \rightarrow K(\text{FlCot} \, R) \) is a left adjoint to the inclusion \( K(\text{FlCot} \, R) \rightarrow K(\text{Flat} \, R) \), and \( \ell^U \) yields the unit morphism of this pair. The kernel of \( \lambda^U \) is the subcategory \( K_{\text{pac}}(\text{Flat} \, R) \) consisting of pure acyclic complexes.

By this theorem, we can conclude that \( \lambda^U \) induces the triangulated equivalence

\[
D(\text{Flat} \, R) \cong K(\text{FlCot} \, R).
\]

It then follows that

\[
K(\text{Proj} \, R) \xrightarrow{\lambda^U} K(\text{FlCot} \, R)
\]

is a triangulated equivalence. As a consequence, we can explicitly describe the link between the two stable categories as follow:

\[
(2.1) \quad \text{GProj} \, R \cong K_{\text{tac}}(\text{Proj} \, R) \xrightarrow{\lambda^U} K_{\text{tac}}(\text{FlCot} \, R) \cong \text{GFlCot} \, R.
\]

See [14] for more details.

3. Purity for weak balanced big Cohen–Macaulay modules

Let \( (R, m, k) \) be a CM local ring. Following [9], we say that an \( R \)-module is \textit{weak balanced big Cohen–Macaulay} if any system of parameters of \( m \) is a weak \( M \)-regular sequence, see [4, Definition 1.1.1]. We denote by \( \text{wbbCM} \, R \) the category of weak balanced big CM modules. It essentially follows from [9, Proposition 2.4] that \( \text{wbbCM} \, R \) is a
definable subcategory of Mod\(R\), that is, a subcategory closed under direct limits, direct product, and pure submodules. Moreover, [9, Theorem B] says that \(\text{wbbCM} \, R\) is the smallest definable subcategory containing maximal CM modules.

We denote by the \(\text{Zg}_R\) the Ziegler spectrum of \(R\), that is, the (small) set of isomorphism classes of indecomposable pure-injective modules, where an \(R\)-module \(P\) is called pure-injective if \(\text{Hom}_R(-, P)\) preserves exactness of pure short exact sequences. The Ziegler spectrum has a natural topology defined through a functor category, see [5, §2.5]. Furthermore, there is a canonical bijection from definable subcategories of Mod\(R\) to closed subsets of \(\text{Zg}_R\), see [17, Corollary 5.1.6].

Suppose that \(R\) is \(m\)-adically complete. Then the Matlis duality implies that all finitely generated \(R\)-modules are pure-injective. Let us denote by \(\text{Zg}_R(\text{wbbCM})\) the closed subset of the Ziegler spectrum of \(R\) consisting of points represented by \(\text{wbbCM}\) (weak balanced big CM) modules. Note that \(\text{Zg}_R(\text{wbbCM})\) contains all indecomposable (maximal) CM modules up to isomorphism. Therefore, this subset provide a natural place to talk on an infinite version of CM representation theory. In this section, we explain that the stable category \(\text{GFlCot} \, R\) could be a suitable tool to study \(\text{Zg}_R(\text{wbbCM})\) when \(R\) is Gorenstein.

We first remark that \(\text{Zg}_R(\text{wbbCM})\) contains trivial points. Let us denote by \(\text{Zg}_R(\text{Flat})\) the (closed) subset of the Ziegler spectrum formed by indecomposable pure-injective flat modules; they are just the indecomposable flat cotorsion modules. Then Enochs’ [6, Theorem] yields a natural bijection

\[
\text{Zg}_R(\text{Flat}) \cong \left\{ \hat{R}_p \mid p \in \text{Spec} \, R \right\},
\]

where \(\hat{R}_p\) stands for the \(p\)-adic completion of \(R_p\). There is an inclusion

\[
\text{Zg}_R(\text{Flat}) \subseteq \text{Zg}_R(\text{wbbCM}),
\]

and the equality holds if and only if \(R\) is regular; this fact essentially follows from [9, Proposition 4.9]. Hence it is natural to take the complement \(\text{Zg}_R(\text{wbbCM}) \setminus \text{Zg}_R(\text{Flat})\).

Now, suppose that \(R\) is Gorenstein. Then we have \(\text{wbbCM} \, R = \text{GFlat} \, R\), see [9, Theorem B]. As we are now interested in pure-injective modules and all pure-injective modules are cotorsion, it is enough to treat \(\text{wbbCM}\) cotorsion modules. Writing

\[
\text{wbbCMC} \, R := \text{wbbCM} \, R \cap \text{Cot} \, R,
\]

we have \(\text{wbbCMC} \, R = \text{GFlCot} \, R\). Taking this fact into account, let us interpret \(\text{GFlCot} \, R\) as the stable category \(\text{wbbCMC} \, R\) of \(\text{wbbCM}\) cotorsion modules modulo flat cotorsion.

Note that all acyclic complexes of flat modules are \(F\)-totally acyclic since \(R\) is now Gorenstein. Hence \(K_{\text{ac}}(\text{FlCot} \, R)\) is nothing but the homotopy category \(K_{\text{ac}}(\text{FlCot} \, R)\) of acyclic complexes of flat cotorsion modules; it is triangulated equivalent to \(\text{wbbCMC} \, R\).

**Example 3.** Let \(R\) be as in Example 1. Consider the mapping cone of the next morphism in \(K_{\text{ac}}(\text{FlCot} \, R)\):

\[
\cdots \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} \cdots
\]

\[
\cdots \xrightarrow{x} R_{(x)} \xrightarrow{x} R_{(x)} \xrightarrow{x} R_{(x)} \xrightarrow{x} \cdots
\]
where the vertical maps are the canonical ones. Through the equivalence \( K_{ac}(\text{FICot } R) \cong \text{wbbCMC } R \), the mapping cone gives a wbbCM module, which is the integral closure \( \bar{R} \).

It essentially follows from Jørgensen’s work \[10\] that the homotopy category \( K_{ac}(\text{Proj } R) \) of acyclic complexes of projective modules is compactly generated, and so is \( \text{GFlCot } R = \text{wbbCMC } R \), see (2.1). Then, thanks to Krause’s work \[11\], we are able to talk about purity in this stable category. The next result is an analogue to the case of quasi-Frobenius rings, see \[11\, \text{Proposition } 1.16\].

**Theorem 4.** An \( R \)-module \( M \in \text{wbbCMC } R \) is pure-injective in \( \text{Mod}R \) if and only if \( M \) is pure-injective in \( \text{wbbCMC } R \).

This result implicitly says that we can at least make a bijection between the Ziegler spectrum of the triangulated category \( \text{wbbCMC } R \) and \( \text{Zg}_R(\text{wbbCM}) \setminus \text{Zg}_R(\text{Flat}) \), although their topological aspects have to be discussed carefully. It should be also remarked that the same statement as the theorem does not hold for Gorenstein-projective modules.

Example 1 deals with a non-isolated singularity \( k[[x, y]]/(x^2) \). To include such a case, we need to consider \( \text{wbbCM} \) modules. However, if \( R \) is a Gorenstein isolated singularity, the equivalences in (2.1) suggests that a smaller stable category is available. We denote by \( \text{bbCM}^\wedge_m R \) the subcategory of \( \text{Mod} R \) formed by \( m \)-adic completions of balanced big CM modules, where an \( R \)-module is called balanced big Cohen–Macaulay if any system of parameters of \( m \) is an \( M \)-regular sequence. When \( R \) is Gorenstein, we can show that \( \text{bbCM}^\wedge_m R \) has a Frobenius structure, where the projective-injective objects are the \( m \)-adic completions of all free \( R \)-modules. Then its stable category \( \text{bbCM}^\wedge_m R \) modulo \( m \)-adic completions of free \( R \)-modules has a triangulated structure. In fact, it is possible to show that there is a triangulated equivalence

\[
K_{ac}(\text{Proj}^\wedge_m R) \cong \text{bbCM}^\wedge_m R,
\]

where \( \text{Proj}^\wedge_m R \) stands for the subcategory of \( \text{Mod} R \) formed by \( m \)-adic completions of free \( R \)-modules.

**Theorem 5.** Let \((R, m)\) be a Gorenstein local ring with an isolated singularity. Then there are triangulated equivalences

\[
\text{GProj } R \cong K_{ac}(\text{Proj } R) \xrightarrow{\Lambda^m} K_{ac}(\text{Proj}^\wedge_m R) \cong \text{bbCM}^\wedge_m R.
\]

In fact, the stable category \( \text{bbCM}^\wedge_m R \) can be also obtained as a full subcategory of \( \text{wbbCMC } R \), and the above theorem implies that they are triangulated equivalent. Therefore, \( \text{Zg}_R(\text{wbbCM}) \setminus \text{Zg}_R(\text{Flat}) \) consists of points represented by indecomposable non-projective balanced big CM modules when \( R \) is a complete Gorenstein local ring with an isolated singularity. This fact leads us to a reasonable setup for a CM version of the result by Tachikawa and Auslander mentioned in the introduction. See \[15\] for more details.

**References**


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