PURE DERIVED CATEGORIES AND WEAK BALANCED BIG COHEN–MACAULAY MODULES

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Abstract. We report a new approach to reach the pure derived category of flat modules over a commutative noetherian ring of finite Krull dimension. Using it, we concretely connect two different stable categories over a Gorenstein ring; the first one is the stable category of Gorenstein-projective modules, and the other is the stable category of Gorenstein-flat cotorsion modules. Although they are triangulated equivalent, we report that the latter has some advantage in terms of pure-injectivity. This advantage along with the notion of weak balanced big Cohen–Macaulay modules naturally leads us to an infinite version of Cohen-Macaulay representation theory.

1. Introduction

A specialist of model theory of modules, Gena Puninski [18] proposed an interesting study of Cohen–Macaulay representations via pure-injectivity. His idea was based on importance of infinitely generated pure-injective modules over artinian rings. As shown by Tachikawa [21, Corollary 9.5] and Auslander [2, Theorem A], a (possibly non-commutative) artinian ring A is of finite representation type if and only if any indecomposable pure-injective module is finitely generated. (See also Auslander [1, Corollary 4.8], Ringel and Tachikawa [19, Corollary 4.4] and Prest [17, §5.3.4].) This fact implies that if A is not of finite representation type, then there exists an infinitely generated pure-injective A-module which is indecomposable. Some of such modules have a role to control behavior of finitely generated modules, see Crawley-Boevey [5]. See also Benson and Krause [3] for importance of pure-injective modules in their context.

The next computation is given by Puninski [18].

Example 1. Let k be an algebraic closed field with $\operatorname{char} k \neq 2$, and set $R = k[[x, y]]/(x^2)$. Indecomposable infinitely generated pure-injective R-modules M with $\operatorname{Hom}_R(k, M) = 0$ are just $R_{(x)}, xR_{(x)}$ and $\bar{R}$ up to isomorphism, where $\bar{R}$ is the integral closure of R in the total quotient ring $R_{(x)}$.

Let $(R, \mathfrak{m}, k)$ be a CM (Cohen–Macaulay) local ring. Puninski meant by a CM R-module an R-module M such that $\operatorname{Ext}^i_R(k, M) = 0$ for $i < \dim R$. To avoid confusion, let us express such modules as "CM" modules. The above example computes all indecomposable infinitely generated "CM" modules over $k[[x, y]]/(x^2)$ such that they are pure-injective.

Unlike artinian rings, R having positive Krull dimension easily admits (trivial) indecomposable pure-injective modules which are infinitely generated. The localization of R at any minimal prime ideal is such an R-module, and it is always a "CM" R-module.

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Moreover, if $\dim R > 1$, then any indecomposable injective module corresponding to a hight-one prime ideal becomes a "CM" R-module, see [18, Remark 10.1]. Therefore we can not simply extend the result of Tachikawa and Auslander. Actually, Puninski [18, Question 10.2] left the next question:

Let R be the formal power series ring in two variables over an algebraic closed field, and let $\mathfrak{m}$ be its maximal ideal. Is any indecomposable pure-injective "CM" R-module M with $M/\mathfrak{m}M \neq 0$ finitely generated?

Although the last condition can avoid a lot of infinitely generated modules, this question might be still unreasonable. One reason is that the vanishing condition of $\mathrm{Ext}^i_R(k, M)$ for $i < \dim R$ need not provide the smallest closed subset of the Ziegler spectrum of R containing all indecomposable finitely generated maximal CM modules up to isomorphism. A better class of modules is formed by *weak balanced big CM modules* in the sense of Holm [9]. His results [9, Theorem B and Proposition 2.4] can imply that the class of these modules corresponds to the smallest closed subset. Moreover, Puninski's "CM" modules agree with Holm's weak balanced big CM modules if R has at most dimension one. Hence we can naturally continue Puninski's work.

However, the smallest closed subset still contains kind of trivial ones; indecomposable flat cotorsion modules. In fact, the closed subset is occupied by them if and if R is regular. Then, in terms of singularity theory, it is very natural to remove them. Sections 2 and 3 are devoted to explain that there is a natural stable category of modules to study the rest part of the closed subset.

2. The pure derived category of flat modules

The stable category of maximal CM modules is a fundamental tool in CM representation theory. It is constructed by identifying two maps whose difference factors through some finitely generated projective module. Over a Gorenstein local ring R, this stable category is triangulated equivalent to the homotopy category $\mathrm{K}_{\mathrm{ac}}(\operatorname{proj} R)$ of acyclic complexes of finitely generated projective modules. Then, the larger category $\mathrm{K}_{\mathrm{ac}}(\operatorname{Proj} R)$ formed by acyclic complexes of arbitrary projective modules and its corresponding stable category consisting of Gorenstein-projective modules look natural places to discuss infinitely generated CM representations. However, focusing on only "modulo projective modules" could lose an important viewpoint of pure-injectivity. This is just because, projective modules need not be pure-injective. In this section, passing through the pure derived category of flat modules, we will arrive at another stable category, which preserves pure-injectivity nicely, and extends the stable category of maximal CM modules, see Section 3.

Let us start with an arbitrary ring A. A complex X of left A-modules is said to be *pure acyclic* if it is acyclic (i.e. exact) and $M \otimes_A X$ is acyclic for any right A-modules. The homotopy category $\mathrm{K}(\operatorname{Flat} A)$ of complexes of flat left A-modules has a full subcategory $\mathrm{K}_{\mathrm{pac}}(\operatorname{Flat} A)$ consisting of pure acyclic complexes. The *pure derived category* $\mathrm{D}(\operatorname{Flat} \mathrm{A})$ *of flat A-modules* is defined as the Verdier quotient category $\mathrm{K}(\operatorname{Flat} A)/\mathrm{K}_{\mathrm{pac}}(\operatorname{Flat} A)$. This category appeared in Neeman's work [16], see also Murfet and Salarian [12]. Neeman proved that the canonical composition $\mathrm{K}(\operatorname{Proj} A) \to \mathrm{K}(\operatorname{Flat} A) \to \mathrm{D}(\operatorname{Flat} A)$ is a triangulated equivalence.

A left A-module M is said to be *cotorsion* if $\operatorname{Ext}^i_A(F, M) = 0$ for any flat left A-module and any positive integer i. The category of cotorsion left A-modules is denoted by $\operatorname{Cot} A$. Moreover, we set $\operatorname{FlCot} A = \operatorname{Flat} A \cap \operatorname{Cot} A$; its objects are called flat cotorsion A-modules. Šťovíček's [20, Corollary 5.8] refining Gillespie's [7, Corollary 4.10] implies that the canonical composition $\mathrm{K}(\operatorname{FlCot} A) \to \mathrm{K}(\operatorname{Flat} A) \to \mathrm{D}(\operatorname{Flat} A)$ is a triangulated equivalence. Consequently, it holds that

$$\mathrm{K}(\operatorname{Proj} A) \cong \mathrm{D}(\operatorname{Flat} A) \cong \mathrm{K}(\operatorname{FlCot} A).$$

Replacement of a complex $X \in \mathrm{K}(\operatorname{Proj} A)$ with $Y \in \mathrm{K}(\operatorname{FlCot} A)$ is given by a pure quasi-isomorphism $X \to Y$, that is, a quasi-isomorphism whose mapping cone is pure acyclic. However, we are not able to understand this replacement in detail by the general theory.

A complex X of projective (resp. flat cotorsion) left A-modules is said to be *totally acyclic* if it is acyclic and $\operatorname{Hom}_A(X, F)$ is acyclic for any projective (flat cotorsion) left A-module F. Moreover, a complex X of flat left A-modules is said to be *F-totally acyclic* if it is acyclic and $E \otimes_A X$ is acyclic for any injective right A-module E. We denote by $\mathrm{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ and $\mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} A)$ the full subcategories of $\mathrm{K}(\operatorname{Proj} A)$ and $\mathrm{K}(\operatorname{FlCot} A)$ formed by totally acyclic complexes respectively. Moreover, let $\mathrm{D}_{\mathrm{Ftac}}(\operatorname{Flat} A)$ be the full subcategory of $\mathrm{D}(\operatorname{Flat} A)$ formed by F-totally acyclic complexes.

Suppose that A is a right coherent ring and any flat left A-module has finite projective dimension. Then any F-totally acyclic complex of projective left A-modules becomes totally acyclic. Similarly, any F-totally acyclic complex of flat cotorsion left A-modules becomes totally acyclic. Therefore, restricting the above triangulated equivalences to F-totally acyclic complexes, we get

$$\mathrm{K}_{\mathrm{tac}}(\operatorname{Proj} A) \cong \mathrm{D}_{\mathrm{Ftac}}(\operatorname{Flat} A) \cong \mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} A).$$

A left A-module is called *Gorenstein-projective* if it is the kernel of the 0th differential map of some totally acyclic complex of projective modules. The category of Gorenstein-projective modules is denoted by $\operatorname{GProj} A$. It is well-known that this category has a Frobenius structure naturally, and hence its stable category $\underline{\operatorname{GProj}}\, A$ modulo projective modules becomes a triangulated category which is triangulated equivalent to $\mathrm{K}_{\mathrm{tac}}(\operatorname{Proj} A)$.

A left A-module is called *Gorenstein-flat* if it is the kernel of the 0th differential map of some F-totally acyclic complex of flat modules. The category of Gorenstein-flat modules is denoted by $\operatorname{GFlat} A$. Its stable category modulo flat modules could be too huge, hence we take a smaller category

$$\operatorname{GFlCot} A := \operatorname{GFlat} A \cap \operatorname{Cot} A,$$

whose objects are Gorenstein-flat cotorsion modules. As observed by Gillespie [8] essentially, this category also has a Frobenius structure, and its stable category $\underline{\operatorname{GFlCot}}\, A$ modulo flat cotorsion modules is triangulated equivalent to $\mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} A)$. Therefore we have

$$\underline{\operatorname{GProj}}\, A \cong \mathrm{K}_{\mathrm{tac}}(\operatorname{Proj} A) \cong \mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} A) \cong \underline{\operatorname{GFlCot}}\, A.$$

We want to understand the replacement of modules between these stable categories. However, as mentioned above, the general theory does not explain this in detail, because it just says that $X \in \mathrm{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ is replaced by some $Y \in \mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} A)$ through a pure quasi-isomorphism $X \to Y$.

Henceforth, R denotes a commutative noetherian ring with finite Krull dimension d. We would like to provide another approach to reach the pure derived category of flat modules over R. Set $U = \operatorname{Spec} R$, and $U_i = \{\mathfrak{p} \in U \mid \dim R/\mathfrak{p} = i\}$. Consider the canonical morphism $\mathrm{id}_{\operatorname{Mod} R} \to \bar{\lambda}^{U_i} = \prod_{\mathfrak{p}\in U_i} \Lambda^{\mathfrak{p}}(R_{\mathfrak{p}} \otimes_R -)$, where $\Lambda^{\mathfrak{p}}$ stands for the p-adic completion functor $\varprojlim_{n\geq 1}(R/\mathfrak{p} \otimes_R -)$ on the category $\operatorname{Mod} R$ of R-modules. Using a standard construction of Čech complexes along with the natural transformations $\mathrm{id}_{\operatorname{Mod} R} \to \bar{\lambda}^{U_i}$, we get a complex of functors:

$$L^{\mathbb{U}} = \left(\prod_{0\leq i\leq d} \bar{\lambda}^{U_i} \longrightarrow \prod_{0\leq i<j\leq d} \bar{\lambda}^{U_j}\bar{\lambda}^{U_i} \longrightarrow \cdots \longrightarrow \bar{\lambda}^{U_d}\cdots\bar{\lambda}^{U_0} \right),$$

where $\mathbb{U}$ stands for the family $\{U_i\}_{0\leq i\leq d}$. To a complex X of R-modules, $L^{\mathbb{U}}$ naturally assigns a double complex $L^{\mathbb{U}}X$ with a canonical morphism $X \to L^{\mathbb{U}}X$. Taking their total complexes, we get a natural chain map $\ell^{\mathbb{U}}X : X \to \mathrm{tot}L^{\mathbb{U}}X$. This construction appeared in the previous work [13] with Yoshino.

For simplicity, let us write $\lambda^{\mathbb{U}} = \mathrm{tot}L^{\mathbb{U}}$, which is an endofunctor on the category $\mathrm{C}(\operatorname{Mod} R)$ of complexes. This can be restricted to an endofunctor on the category $\mathrm{C}(\operatorname{Flat} R)$ of complexes of flat modules. More precisely, it factors through the inclusion from the category $\mathrm{C}(\operatorname{FlCot} R)$ of complexes of flat cotorsion modules into $\mathrm{C}(\operatorname{Flat} R)$:

$$\mathrm{C}(\operatorname{Flat} R) \xrightarrow{\lambda^{\mathbb{U}}} \mathrm{C}(\operatorname{FlCot} R) \xrightarrow{\mathrm{inc}} \mathrm{C}(\operatorname{Flat} R).$$

The above sequence naturally induces a sequence of triangulated functors on homotopy categories:

$$\mathrm{K}(\operatorname{Flat} R) \xrightarrow{\lambda^{\mathbb{U}}} \mathrm{K}(\operatorname{FlCot} R) \xrightarrow{\mathrm{inc}} \mathrm{K}(\operatorname{Flat} R).$$

Theorem 2. *The triangulated functor* $\lambda^{\mathbb{U}} : \mathrm{K}(\operatorname{Flat} R) \to \mathrm{K}(\operatorname{FlCot} R)$ *is a left adjoint to the inclusion* $\mathrm{K}(\operatorname{FlCot} R) \to \mathrm{K}(\operatorname{Flat} R)$*, and* $\ell^{\mathbb{U}}$ *yields the unit morphism of this pair. The kernel of* $\lambda^{\mathbb{U}}$ *is the subcategory* $\mathrm{K}_{\mathrm{pac}}(\operatorname{Flat} R)$ *consisting of pure acyclic complexes.*

By this theorem, we can conclude that $\lambda^{\mathbb{U}}$ induces the triangulated equivalence

$$\mathrm{D}(\operatorname{Flat} R) \cong \mathrm{K}(\operatorname{FlCot} R).$$

It then follows that

$$\mathrm{K}(\operatorname{Proj} R) \xrightarrow{\lambda^{\mathbb{U}}} \mathrm{K}(\operatorname{FlCot} R)$$

is a triangulated equivalence. As a consequence, we can explicitly describe the link between the two stable categories as follow:

$$\underline{\operatorname{GProj}} R \cong \mathrm{K}_{\mathrm{tac}}(\operatorname{Proj} R) \underset{\cong}{\xrightarrow{\lambda^{\mathbb{U}}}} \mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} R) \cong \underline{\operatorname{GFlCot}} R. \tag{2.1}$$

See [14] for more details.

3. Purity for weak balanced big Cohen–Macaulay modules

Let $(R, \mathfrak{m}, k)$ be a CM local ring. Following [9], we say that an R-module is *weak balanced big Cohen–Macaulay* if any system of parameters of $\mathfrak{m}$ is a weak M-regular sequence, see [4, Definition 1.1.1]. We denote by $\operatorname{wbbCM} R$ the category of weak balanced big CM modules. It essentially follows from [9, Proposition 2.4] that $\operatorname{wbbCM} R$ is a *definable* subcategory of $\operatorname{Mod} R$, that is, a subcategory closed under direct limits, direct product, and pure submodules. Moreover, [9, Theorem B] says that $\operatorname{wbbCM} R$ is the smallest definable subcategory containing maximal CM modules.

We denote by the Zg_R the *Ziegler spectrum* of R, that is, the (small) set of isomorphism classes of indecomposable pure-injective modules, where an R-module P is called *pure-injective* if $\operatorname{Hom}_R(-, P)$ preserves exactness of pure short exact sequences. The Ziegler spectrum has a natural topology defined through a functor category, see [5, §2.5]. Furthermore, there is a canonical bijection from definable subcategories of $\operatorname{Mod} R$ to closed subsets of Zg_R, see [17, Corollary 5.1.6].

Suppose that R is $\mathfrak{m}$-adically complete. Then the Matlis duality implies that all finitely generated R-modules are pure-injective. Let us denote by $\operatorname{Zg}_R(\mathrm{wbbCM})$ the closed subset of the Ziegler spectrum of R consisting of points represented by wbbCM (weak balanced big CM) modules. Note that $\operatorname{Zg}_R(\mathrm{wbbCM})$ contains all indecomposable (maximal) CM modules up to isomorphism. Therefore, this subset provide a natural place to talk on an infinite version of CM representation theory. In this section, we explain that the stable category $\underline{\mathrm{GFlCot}}\, R$ could be a suitable tool to study $\operatorname{Zg}_R(\mathrm{wbbCM})$ when R is Gorenstein.

We first remark that $\operatorname{Zg}_R(\mathrm{wbbCM})$ contains trivial points. Let us denote by $\operatorname{Zg}_R(\mathrm{Flat})$ the (closed) subset of the Ziegler spectrum formed by indecomposable pure-injective flat modules; they are just the indecomposable flat cotorsion modules. Then Enochs' [6, Theorem] yields a natural bijection

$$\operatorname{Zg}_R(\mathrm{Flat}) \cong \left\{ \widehat{R_\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R \right\},$$

where $\widehat{R_\mathfrak{p}}$ stands for the $\mathfrak{p}$-adic completion of $R_\mathfrak{p}$. There is an inclusion

$$\operatorname{Zg}_R(\mathrm{Flat}) \subseteq \operatorname{Zg}_R(\mathrm{wbbCM}),$$

and the equality holds if and only if R is regular; this fact essentially follows from [9, Proposition 4.9]. Hence it is natural to take the complement $\operatorname{Zg}_R(\mathrm{wbbCM}) \backslash \operatorname{Zg}_R(\mathrm{Flat})$.

Now, suppose that R is Gorenstein. Then we have $\operatorname{wbbCM} R = \operatorname{GFlat} R$, see [9, Theorem B]. As we are now interested in pure-injective modules and all pure-injective modules are cotorsion, it is enough to treat wbbCM cotorsion modules. Writing

$$\operatorname{wbbCMC} R := \operatorname{wbbCM} R \cap \operatorname{Cot} R,$$

we have $\operatorname{wbbCMC} R = \operatorname{GFlCot} R$. Taking this fact into account, let us interpret $\underline{\mathrm{GFlCot}}\, R$ as the stable category $\underline{\mathrm{wbbCMC}}\, R$ of wbbCM cotorsion modules modulo flat cotorsion.

Note that all acyclic complexes of flat modules are F-totally acyclic since R is now Gorenstein. Hence $\mathrm{K}_{\mathrm{tac}}(\operatorname{FlCot} R)$ is nothing but the homotopy category $\mathrm{K}_{\mathrm{ac}}(\operatorname{FlCot} R)$ of acyclic complexes of flat cotorsion modules; it is triangulated equivalent to $\underline{\mathrm{wbbCMC}}\, R$.

Example 3. Let R be as in Example 1. Consider the mapping cone of the next morphism in $\mathrm{K}_{\mathrm{ac}}(\operatorname{FlCot} R)$:

$$\begin{tikzcd}
\cdots \arrow[r, "x"] & R \arrow[r, "x"] \arrow[d] & R \arrow[r, "x"] \arrow[d] & R \arrow[r, "x"] \arrow[d] & \cdots \\
\cdots \arrow[r, "x"] & R_{(x)} \arrow[r, "x"] & R_{(x)} \arrow[r, "x"] & R_{(x)} \arrow[r, "x"] & \cdots
\end{tikzcd}$$

where the vertical maps are the canonical ones. Through the equivalence $\mathrm{K_{ac}}(\mathrm{FlCot}\, R) \cong \underline{\mathrm{wbbCMC}}\, R$, the mapping cone gives a wbbCM module, which is the integral closure $\bar{R}$.

It essentially follows from Jørgensen's work [10] that the homotopy category $\mathrm{K_{ac}}(\mathrm{Proj}\, R)$ of acyclic complexes of projective modules is compactly generated, and so is $\underline{\mathrm{GFlCot}}\, R = \underline{\mathrm{wbbCMC}}\, R$, see (2.1). Then, thanks to Krause's work [11], we are able to talk about purity in this stable category. The next result is an analogue to the case of quasi-Frobenius rings, see [11, Proposition 1.16].

Theorem 4. *An R-module $M \in \mathrm{wbbCMC}\, R$ is pure-injective in $\mathrm{Mod}R$ if and only if M is pure-injective in $\underline{\mathrm{wbbCMC}}\, R$.*

This result implicitly says that we can at least make a bijection between the Ziegler spectrum of the triangulated category $\underline{\mathrm{wbbCMC}}\, R$ and $\mathrm{Zg}_R(\mathrm{wbbCM})\backslash\mathrm{Zg}_R(\mathrm{Flat})$, although their topological aspects have to be discussed carefully. It should be also remarked that the same statement as the theorem does not hold for Gorenstein-projective modules.

Example 1 deals with a non-isolated singularity $k[[x, y]]/(x^2)$. To include such a case, we need to consider wbbCM modules. However, if R is a Gorenstein isolated singularity, the equivalences in (2.1) suggests that a smaller stable category is available. We denote by $\mathrm{bbCM}^{\wedge}_{\mathfrak{m}}\, R$ the subcategory of $\mathrm{Mod}\, R$ formed by $\mathfrak{m}$-adic completions of balanced big CM modules, where an R-module is called *balanced big Cohen–Macaulay* if any system of parameters of $\mathfrak{m}$ is an M-regular sequence. When R is Gorenstein, we can show that $\mathrm{bbCM}^{\wedge}_{\mathfrak{m}}\, R$ has a Frobenius structure, where the projective-injective objects are the $\mathfrak{m}$-adic completions of all free R-modules. Then its stable category $\underline{\mathrm{bbCM}^{\wedge}_{\mathfrak{m}}}\, R$ modulo $\mathfrak{m}$-adic completions of free R-modules has a triangulated structure. In fact, it is possible to show that there is a triangulated equivalence

$$\mathrm{K_{ac}}(\mathrm{Proj}^{\wedge}_{\mathfrak{m}}\, R) \cong \underline{\mathrm{bbCM}^{\wedge}_{\mathfrak{m}}}\, R,$$

where $\mathrm{Proj}^{\wedge}_{\mathfrak{m}}\, R$ stands for the subcategory of $\mathrm{Mod}\, R$ formed by $\mathfrak{m}$-adic completions of free R-modules.

Theorem 5. *Let $(R, \mathfrak{m})$ be a Gorenstein local ring with an isolated singularity. Then there are triangulated equivalences*

$$\underline{\mathrm{GProj}}\, R \cong \mathrm{K_{ac}}(\mathrm{Proj}\, R) \xrightarrow[\cong]{\Lambda^{\mathfrak{m}}} \mathrm{K_{ac}}(\mathrm{Proj}^{\wedge}_{\mathfrak{m}}\, R) \cong \underline{\mathrm{bbCM}^{\wedge}_{\mathfrak{m}}}\, R.$$

In fact, the stable category $\underline{\mathrm{bbCM}^{\wedge}_{\mathfrak{m}}}\, R$ can be also obtained as a full subcategory of $\underline{\mathrm{wbbCMC}}\, R$, and the above theorem implies that they are triangulated equivalent. Therefore, $\mathrm{Zg}_R(\mathrm{wbbCM})\backslash\mathrm{Zg}_R(\mathrm{Flat})$ consists of points represented by indecomposable non-projective balanced big CM modules when R is a complete Gorenstein local ring with an isolated singularity. This fact leads us to a reasonable setup for a CM version of the result by Tachikawa and Auslander mentioned in the introduction. See [15] for more details.

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